Conditional representation stability, classification of $*$-homomorphisms, and relative eta invariants

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Abstract

A quasi-representation of a group is a map from the group into a matrix algebra (or similar object) that approximately satisfies the relations needed to be a representation. Work of many people starting with Kazhdan and Voiculescu, and recently advanced by Dadarlat, Eilers-Shulman-Sørensen and others, has shown that there are topological obstructions to approximating unitary quasi-representations of groups by honest representations, where 'approximation' is understood to be with respect to the operator norm.

The purpose of this paper is to explore whether approximation is possible if the known obstructions vanish, partially generalizing work of Gong-Lin and Eilers-Loring-Pedersen for the free abelian group of rank two, and the Klein bottle group. We show that this is possible, at least in a weak sense, for some 'low-dimensional' groups including fundamental groups of closed surfaces, certain Baumslag-Solitar groups, free-by-cyclic groups, and many fundamental groups of three manifolds.

The techniques used in the paper are K-theoretic: they have their origin in Baum-Connes-Kasparov type assembly maps, and in the Elliott program to classify C^*-algebras; Kasparov's bivariant KK-theory is a crucial tool. The key new technical ingredients are: a stable uniqueness theorem in the sense of Dadarlat-Eilers and Lin that works for non-exact C^*-algebras; and an analysis of maps on K-theory with finite coefficients in terms of the relative eta invariants of Atiyah-Patodi-Singer. Despite the proofs going through K-theoretic machinery, the main theorems can be stated in elementary terms that do not need any K-theory.

Contents

1 Introduction

The goal of this paper is to study an instance of the following general question: if one has an approximate solution to an equation, must it be close to an actual solution?

We study this in the context of unitary representations of discrete groups, and where the approximations take place in the operator norm. Starting with work of Voiculescu [134] and Kazhdan [82], it is known that there are topological obstructions to a positive answer. The purpose of the current paper is to explore what happens when the known obstructions vanish: specifically, is vanishing sufficient for an approximation by an actual solution to exist?

1.1 Almost commuting unitaries

The main topic of this paper is finite dimensional unitary representations, and approximate representations, of discrete groups. However, we will start with a more basic question about matrices as that motivates our main questions and results; informally, it asks if any pair of approximately commuting unitary matrices can be approximated by an actually commuting pair.

All norms in this paper are operator norms on spaces of bounded operators on (possibly finite dimensional) Hilbert spaces.

Question 1.1. *For any $\epsilon > 0$, does there exist $\delta > 0$ with the following property?*

For any n[1], if $u, v \in M_n(\mathbb{C})$ are unitary matrices satisfying

$$\|uv - vu\| < \delta,$$

then there are unitary matrices $u', v' \in M_n(\mathbb{C})$ satisfying

$$\|u' - u\| < \epsilon, \quad \|v' - v\| < \epsilon, \quad \text{and} \quad u'v' = v'u'.$$

The question is completely answered by the following result. To state it, we need some notation. Let $u, v \in M_n(\mathbb{C})$ be unitary matrices such that $\|uv - vu\| < 1$. The function

$$[0,1] \ni t \mapsto \det\big(t + (1-t)uvu^{-1}v^{-1}\big) \tag{1}$$

then defines a path in the complex numbers that starts and ends at 1, and avoids 0. Hence we may define $w(u, v)$ to be the winding number of this path.

Theorem 1.2 (Several contributors - see below)**.** *(i) There is $\epsilon > 0$ such that for any n, if $u, v \in M_n(\mathbb{C})$ are unitary matrices such that $\|uv - vu\| < 1$ and $w(u, v) \neq 0$, then there do not exist unitary matrices $u', v' \in M_n(\mathbb{C})$ with $\|u' - u\| < \epsilon$, $\|v' - v\| < \epsilon$ and $u'v' = v'u'$.*

(ii) For any $\epsilon > 0$ there exists $\delta > 0$ such that for any n, if $u, v \in M_n(\mathbb{C})$ are unitary matrices satisfying $\|uv - vu\| < \delta$ and $w(u, v) = 0$, then there are unitary matrices u' and v' satisfying $\|u' - u\| < \epsilon$, $\|v' - v\| < \epsilon$ and $u'v' = v'u'$.

Moreover, for any $\delta > 0$ and $k \in \mathbb{Z}$ there exist n and unitary matrices $u, v \in M_n(\mathbb{C})$ with $\|uv - vu\| < \delta$ and $w(u, v) = k$.

In brief, Theorem 1.2 says there is a robust integer-valued topological invariant $w(u, v)$ that completely determines whether an almost solution of the equation "$uv = vu$" is close to an actual solution.

Part (i) has its roots in work of Voiculescu [134]. The interpretation in terms of winding numbers comes from work of Loring [99] and of Exel

[1]The order of quantifiers is crucial: if δ is allowed to depend on n, the answer is different.

and Loring [61] in this context, and also of Kazhdan in a slightly different setting [82]. See also [59] for a far-reaching recent approach to this and related questions.

Part (ii) is due independently to Gong and Lin [67, Corollary M3], and to Eilers, Loring, and Pedersen [55, Corollary 6.15]. It seems less well-known than part (i). The reader might compare it to [7, Main Theorem on page 746] (respectively, [66, Theorem 2]): this establishes a similar result for almost commuting elements of permutation groups (respectively, almost commuting unitaries with respect to the normalized Hilbert-Schmidt norm) that holds without any analogue of the winding number condition.

Now, part (i) of Theorem 1.2 has been generalized extensively. Indeed, it can be reframed as asking whether an approximate unitary representation of the group $\mathbb{Z}^2$ is close to an actual representation. Generalized from here to other discrete groups, it becomes a fundamental statement about 'stability' of group representations: see for example [6] and [131] for surveys. Generalizations of part (ii) of Theorem 1.2 have not received so much attention: it is the aim of this paper to say something in this direction.

1.2 Representation stability

We now reformulate Theorem 1.2 in terms of representation theory and in the process bring a little more topology into play; this needs some terminology. Throughout, H denotes a Hilbert space, $\mathcal{B}(H)$ the bounded operators on H, and $\mathcal{B}(H)_1$ the closed unit ball of $\mathcal{B}(H)$. We will mainly be interested in the case $H = \mathbb{C}^n$ so that $\mathcal{B}(H) = M_n(\mathbb{C})$.

Definition 1.3. Let Γ be a discrete group, let S be a subset of Γ, and let $\epsilon > 0$. An *(S,ϵ)-representation* of Γ is a unital map

$$\pi : \Gamma \to \mathcal{B}(H)_1$$ [2]

such that

$$\|\pi(s)\pi(t) - \pi(st)\| < \epsilon$$

for all $s, t \in S$.

If we do not want to specify the pair (S, ϵ), we will just say that ϕ is a *quasi-representation*[3].

[2] It is probably more common in the literature to force a quasi-representation ϕ to have values in the group of unitary operators on the Hilbert space. The two definitions are equivalent up to an approximation: see Lemma 2.1 below. The extra flexibility allowed by our definition is important to us mainly when we need to consider ucp quasi-representations as in Definition 1.6 below: see Proposition 2.2, part (iv) below, which says that the theory of unitary-valued ucp quasi-representations is essentially trivial.

[3] A quasi-representation is therefore just a unital map $\Gamma \to \mathcal{B}(H)_1$. It may seem a bit silly to introduce terminology for this; the point is to emphasize that we are currently thinking of the map as an approximate representations.

Definition 1.4. A group Γ is *stable*[4] if for any finite subset S of Γ and $\epsilon > 0$ there exists a finite subset T of Γ and $\delta > 0$ such that if $\phi : \Gamma \to M_n(\mathbb{C})_1$ is a (T, δ)-representation in the sense of Definition 1.3, then there exists a unitary representation $\pi : \Gamma \to M_n(\mathbb{C})$ such that

$$\|\phi(s) - \pi(s)\| < \epsilon$$

for all $s \in S$.

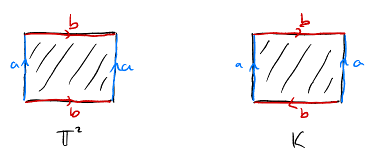
If P is a property of quasi-representations, then we say that Γ is *stable, conditional on* P if for any finite subset S of Γ and $\epsilon > 0$ there exists a finite subset T of Γ and $\delta > 0$ such that if $\phi : \Gamma \to M_n(\mathbb{C})_1$ is a (T, δ)-representation satisfying P, then there exists a unitary representation $\pi : \Gamma \to M_n(\mathbb{C})$ such that

$$\|\phi(s) - \pi(s)\| < \epsilon$$

for all $s \in S$.

Group theoretic analogues of Definition 1.4 (with more general 'targets' than matrix algebras) can be found in [7, Definition 3.1] and [34, Definition 1.9]. Purely C^*-algebraic analogues (with more general 'domains' than groups), can be found in [54, Definition 2.2.7 and Proposition 2.2.9]. See [56, Sections 2.1 and 2.2] for a more recent survey of these C^*-algebraic notions and their translation to group theory; in particular, see [56, Proposition 2.16] for a reformulation in terms of generators and relations for finitely presented groups. Compare also [53, Section 5.2] for a purely C^*-algebraic analogue of conditional stability, which the authors of [53] call *stability with contingencies*.

Switching to representation theory from the matrices of Theorem 1.2, let us state another illustrative result that was a major motivation for us. Let $\mathbb{Z}^2$ be the fundamental group of the two-torus $\mathbb{T}^2$, and let $\mathbb{Z} \rtimes \mathbb{Z}$ denote the fundamental group of the Klein bottle K. These spaces can be illustrated as CW complexes in a standard way as follows



leading to the presentations $\mathbb{Z}^2 = \langle a, b \mid aba^{-1}b^{-1}\rangle$ and $\mathbb{Z} \rtimes \mathbb{Z} = \langle a, b \mid aba^{-1}b\rangle$ defined by reading off the edges around the 2-disk. The fact that the relation "$aba^{-1}b^{-1}$" is a commutator leads to the fact that $H_2(\mathbb{T}^2) = \mathbb{Z}$; as $aba^{-1}b$ is not a commutator, $H_2(K) = 0$. Relatedly, the fact that "$aba^{-1}b^{-1}$" is a commutator is important in well-definedness of

[4] In some other references (for example, [56] or [43]), the property defined here is called *matricial* stability. We drop the word "matricial" as we will not study any other kinds.

the winding number of the path in line (1), while the fact that "$aba^{-1}b$" is not a commutator means that the path defined by

$$[0,1] \ni t \mapsto \det\big(t + (1-t)uvu^{-1}v\big)$$

for unitaries $u, v \in M_n(\mathbb{C})$ does not usually have a well-defined winding number (even if $uvu^{-1}v$ is very close to 1), as it does not start and end at the same place. These connections to (co)homology are already noted in the original papers of Voiculescu [134, page 431] and Kazhdan [82, page 321].

One has in fact the following result.

Theorem 1.5 (Gong-Lin, Eilers-Loring-Pedersen). *(i) The group $\mathbb{Z}^2$ is stable, conditional on vanishing of the winding number invariant $w(u, v)$.*

(ii) The group $\mathbb{Z} \rtimes \mathbb{Z}$ is stable.

Part (i) is just a restatement of Theorem 1.2, part (ii). Part (ii) is due to Eilers, Loring, and Pedersen [54, Theorem 4.3.1 and Corollary 8.2.2; see also table on page 139].

Our original motivation for writing this paper was to establish an analogue of Theorem 1.5 for higher genus surfaces. Here a direct analogue of Theorem 1.2 part (i) is known (and essentially due to Kazhdan [82]), but part (ii) was open.

Our main results give partial generalizations of Theorem 1.5 to several interesting classes of groups which contain the fundamental groups of the torus and Klein bottle: for example, our work applies to fundamental groups of higher genus closed surfaces (the original motivation), free-by-cyclic groups, and certain Baumslag-Solitar groups; all three of these classes contain $\mathbb{Z}^2$ and $\mathbb{Z} \rtimes \mathbb{Z}$. Moreover, all of the groups in these classes other than $\mathbb{Z}^2$ and $\mathbb{Z} \rtimes \mathbb{Z}$ are non-amenable, which requires new arguments (and is why we can only get partial generalizations). We set up the necessary terminology in the next section.

1.3 Concrete results

We need a special class of quasi-representation. This is classical, going back (at least) to Naimark [108].

Definition 1.6. An (S, ϵ)-representation ϕ is *unital*[5] *completely positive* (ucp) if for any finite subset F of Γ any any finite subset of $(z_g)_{g \in F}$ of $\mathbb{C}$ indexed by F, the operator

$$\sum_{g,h \in F} \overline{z_g} z_h \phi(g^{-1}h) \in \mathcal{B}(H)$$

[5] The word "unital" is redundant – all quasi-representations preserve identities for us – but we keep as it is the standard convention in the C^*-algebra literature.

is positive.

We next recall a notion of weak stability[6] due to Dadarlat [43, page 2].

Definition 1.7. A group Γ is *weakly (ucp) stable* if for any finite subset S of Γ and $\epsilon > 0$ there exists a finite subset T of Γ and $\delta > 0$ such that if $\phi : \Gamma \to M_n(\mathbb{C})_1$ is a (ucp) (T, δ)-representation, then there exist unitary representations $\theta : \Gamma \to M_k(\mathbb{C})$ and $\pi : \Gamma \to M_{n+k}(\mathbb{C})$ such that

$$\|(\phi(s) \oplus \theta(s)) - \pi(s)\| < \epsilon$$

for all $s \in S$.

If P is a property of quasi-representations, then we say that Γ is *weakly (ucp) stable, conditional on* P if for any finite subset S of Γ and $\epsilon > 0$ there exists a finite subset T of Γ and $\delta > 0$ such that if $\phi : \Gamma \to M_n(\mathbb{C})_1$ is a (ucp) (T, δ)-representation satisfying P, then there exist unitary representations $\theta : \Gamma \to M_k(\mathbb{C})$ and $\pi : \Gamma \to M_{n+k}(\mathbb{C})$ and a unitary $u \in M_{n+k}(\mathbb{C})$ such that

$$\|(\phi(s) \oplus \theta(s)) - \pi(s)\| < \epsilon$$

for all $s \in S$.

The method of proof of Theorem 1.2 part (i) gives that $\Gamma = \mathbb{Z}^2$ is not even weakly stable. There have been many other 'no go' results along these lines: see for example [56, 43, 65, 15].

Let us now state some of our main results.

Theorem 1.8. *(i) Let*

$$\Gamma = \left\langle a_1, ..., a_g, b_1, ..., b_g \;\middle|\; \prod_{i=1}^{g} [a_i, b_i] \right\rangle$$

be the fundamental group of a closed orientable surface of genus g. For any quasi-representation $\phi : \Gamma \to M_n(\mathbb{C})$ such that

$$\left\| \prod_{i=1}^{g} [\phi(a_i), \phi(b_i)] - 1 \right\| < 1,$$

define $w(\phi) \in \mathbb{Z}$ to be the winding number of the path

$$[0,1] \ni t \mapsto det\left(t + (1-t) \prod_{i=1}^{g} [\phi(a_i), \phi(b_i)] \right). \tag{2}$$

Then Γ is weakly ucp stable, conditional on $w(\phi) = 0$.

[6] There are (at least) three different notions of "weak stability" in related contexts that we are aware of, and mention here to help avoid confusion: (i) the definition of Eilers, Shulman, and Sørensen from [56, Definition 2.10], which has its origin in C^*-algebra theory (see [100, Section 4.1]) and is closer to what we call "stability" in Definition 1.4; (ii) the definition used by Arzhantseva and Păunsecu in [7, Definition 7.1], which considers approximate representations that asymptotically take non-trivial elements a bounded distance away from the identity; and (iii) the definition of "weak ucp stability" used by Dogon in [51, Definition 1.5], which is closer to the use we get out of the so-called LLP in Proposition 2.6 below.

(ii) Let

$$\Gamma = \left\langle a_1, ..., a_{g+1} \;\middle|\; \prod_{i=1}^{g+1} a_i^2 \right\rangle$$

be the fundamental group a closed non-orientable surface of genus g. Then Γ is weakly ucp stable.

Theorem 1.8 should be compared to the result of Lazarovich, Levit, and Minsky [94] on quasi-representations of surface groups with values in permutation groups. In the language of [94, Section 1], the main result of that paper shows that surface groups are *flexibly stable* for maps to permutation groups. Apart from the focus on permutation representations, the result of [94] differs from ours in three important ways: first it is 'absolute', i.e. works for all permutation quasi-representations without topological conditions; second the notion of "flexible stability" from [94, Section 1] is different from "weak stability" as in Definition 1.7 in that flexible stability only allows block sum with an auxiliary trivial representation, and weak stability allows block sum with any representation; and third, the notion of flexible stability used in [94] gives control over the size of the auxiliary trivial representation that one needs to add, and Theorem 1.8 gives no control on the size of the auxiliary representation σ appearing in Definition 1.7.

This should also be compared to the use of the term "flexible stability" by Becker and Lubotzky in [19, Section 4.4][7]. The definition Becker and Lubotzky give of "flexible stability" allows one to take block sum with an auxiliary *quasi*-representation that is of controlled size relative to the original representation. On the other hand, the notion Becker and Lubotzky call *very flexible stability* drops the control on the size of the auxiliary summand; as this auxiliary summand is allowed to be a quasi-representation, this should be considered a weaker analogue of the condition we call "weak stability" as in Definition 1.7.

The winding number invariant appearing in Theorem 1.8 is due to Kazhdan [82]. Dadarlat gives alternative interpretations in terms of index theory and Chern classes in [41, Section 4]; these results, and Dadarlat's more general results in [44], are crucial for the proof of Theorem 1.8, and of the related results we give below. For these theorems, we note that if $c \in H_2(\Gamma)$ is a group homology class and ϕ is a (suitable) quasi-representation, then Dadarlat [44] shows how to define a winding number invariant $w(c, \phi)$: see Definition 7.2 and Theorem 7.3 below for details.

Now, Theorem 1.8 should be thought of as a partial generalization of Theorem 1.5 to a class of non-amenable groups. Here are two other partial generalizations.

Recall first that a *free-by-cyclic group* is a group of the form $F \rtimes_\phi \mathbb{Z}$, where F is a finite rank free group, and $\mathbb{Z}$ acts on F by an automorphism

[7] Thanks to Tatiana Shulman for explaining this to me.

ϕ; note that this class of groups includes the torus and Klein bottle groups of Theorem 1.5 as the two special cases where $F = \mathbb{Z}$ and ϕ is either the trivial or non-trivial automorphism.

Theorem 1.9. *Let $\Gamma = F \rtimes_\phi \mathbb{Z}$ be a free-by-cyclic group.*

Let n be the rank of F, and let $\phi_* : \mathbb{R}^n \to \mathbb{R}^n$ be the map induced by ϕ on the first real homology group. Then $H_2(\Gamma)$ is free abelian, with rank equal to the multiplicity m of the eigenvalue 1 of ϕ_*; choose a basis $c_1, ..., c_m$ for $H_2(\Gamma)$.

The group Γ is weakly stable, conditional on vanishing of the winding number invariants $w(c_i, \phi)$ for all $i \in \{1, ..., m\}$. In particular, if ϕ_* does not have 1 as as an eigenvalue, then Γ is weakly stable.

Here is a third partial generalization to some Baumslag-Solitar groups. Recall that the *Baumslag-Solitar group* $BS(n, m)$ is defined by the presentation $\langle a, b \mid ab^n a^{-1} b^{-m} \rangle$; note that the torus and Klein bottle groups are $BS(1, 1)$ and $BS(1, -1)$ respectively. If neither $|n|$ nor $|m|$ are one, then $BS(n, m)$ is non-amenable. In the non-amenable case, if $|n| \neq |m|$ then $BS(n, m)$ is not residually finite, and a (weak) stability result seems unlikely. If, however, $|n| = |m|$ we have the following result.

Theorem 1.10. *(i) If $\Gamma = BS(n, n)$, then $H_2(\Gamma) \cong \mathbb{Z}$, and there is an associated winding number invariant $w(c, \phi)$ for a choice of generator c of this group. The group Γ is weakly ucp stable, conditional on vanishing of $w(c, \phi)$.*

(ii) The group $BS(n, -n)$ is weakly ucp stable.

Let us conclude this section with a theorem for 3-manifold groups. To state this, let Γ be the fundamental group of an aspherical 3-manifold.

Theorem 1.11. *Let Γ be the fundamental group of a closed, aspherical*[8] *three-manifold M, and assume moreover that either Γ is solvable, or M is hyperbolic. Choose a decomposition $H_2(M) = F \oplus T$ into the sum of a free subgroup and a torsion subgroup*[9]*, choose a basis $c_1, ..., c_m$ for the free summand, and let $w(c_i, \phi)$ be the associated winding number invariants. Then Γ is weakly ucp stable, conditional on $w(c_i, \phi) = 0$ for $i \in \{1, ..., m\}$.*

That the vanishing conditions in Theorems 1.8, 1.9, 1.10, and 1.11 are *necessary* for weak stability is well-known. This was observed by multiple authors for surface groups starting (essentially) with the work of Kazhdan [82], and more recently made explicit by Dadarlat [41, Section 4] and Eilers, Shulman, and Sørensen [56, Section 4.3]. For the statements about three manifolds, $BS(n, n)$ and free-by-cyclic groups, necessity of the vanishing conditions follows from the main result of [44], and also

[8]Recall that a manifold or CW complex M is *aspherical* if it has contractible universal cover, or equivalently if $\pi_n(M) = 0$ for $n > 1$.

[9]If M is orientable, T is trivial, but not in general.

from [56, Theorem 4.10] for $BS(n,n)$. The point of the theorems above is therefore to establish *sufficiency*.

The reader might have noted that the word "ucp" is present in Theorems 1.8, 1.10, and 1.11, but missing from Theorem 1.9, which is thus stronger in some sense. This is due to free-by-cyclic groups satisfying a property called the *local lifting property* (LLP); in this context, it implies that general quasi-representations can be approximated by ucp quasi-representations in an appropriate sense. If the LLP were to hold for the relevant groups then the word "ucp" would not be necessary in Theorems 1.8, 1.10, and 1.11 either.

One cannot remove the word "weakly" from the conclusion of Theorem 1.11: indeed, Theorem 1.11 applies to $\mathbb{Z}^3$, but this group is known not to be conditionally stable with respect to the winding number vanishing condition in Theorem 1.11 by [67, Theorem 4.2]; compare also [56, Theorem 3.13] on crystallographic groups in this regard. We do not know if one can remove the word "weakly" from the conclusions of Theorems 1.8, 1.9, or 1.10.

1.4 C^*-algebra K-theory and stable uniqueness

We stated our results in the previous section in terms of winding numbers as this seemed more elementary and explicit. Underlying these winding numbers are K-theory classes, however, and it is probably fair to say that K-theory is more directly relevant to the problem[10]: this is evidenced by Loring's analysis [99] of Voiculescu's example [134] in terms of Bott periodicity, and computations showing that one can deduce Bott periodicity from properties of almost commuting matrices [138].

We will thus work with K-theory of C^*-algebras, as well as the dual K-homology theory. See [122] for background on C^*-algebra K-theory, and [74] or [22] for background on (K-theory and) K-homology.

Recall that a C^*-algebra is a norm-closed and $*$-closed subalgebra of the $*$-algebra of bounded operator on a Hilbert space; C^*-algebras will be important to us mainly as things we can take the K-theory of. Let $C^*(\Gamma)$ be the maximal, or full, group C^*-algebra of Γ, i.e. the completion of the complex group algebra $\mathbb{C}[\Gamma]$ in the norm it inherits as from the supremum of the norms in all unitary representations of $\mathbb{C}[\Gamma]$; $C^*(\Gamma)$ has the universal property that any unitary representation of Γ extends uniquely by linearity and continuity to a $*$-representation of $C^*(\Gamma)$. Let $K_0(C^*(\Gamma))$ be the (algebraic) K_0 group of $C^*(\Gamma)$, and let $K_0(\mathbb{C})$ be the K_0 group of the complex numbers; by Morita invariance, this agrees with $K_0(M_n(\mathbb{C}))$ for all n.

Now, given a quasi-representation ϕ, we aim to:

[10] Our opinions (biases?) on this are influenced by [9, page 245].

(i) show that it in some sense defines an element

$$\phi_* \in \mathrm{Hom}(K_0(C^*(\Gamma)), K_0(\mathbb{C}))\ ;$$

(ii) show that this element is equivalent to an honest representation π (in an appropriate weak sense) as long as

$$\phi_* = \pi_* \quad \text{in} \quad \mathrm{Hom}(K_0(C^*(\Gamma), K_0(\mathbb{C})).$$

Such theorems exist in the C^*-algebra literature as part of the classification program for simple nuclear C^*-algebras: see [57] for the set-up of this program, and [137] for a survey of recent progress. They are called *stable*[11] *uniqueness theorems* after work of Lin [96] and Dadarlat-Eilers [45, Section 4]. There are numerous technicalities involved here: indeed, to establish (i) one seems to need that ϕ is ucp; and to deal with (ii), K-theory with integral coefficients is not enough, as one instead needs to work with integer coefficients and with all finite coefficients at once.

Now, if we 'only' wanted to work with amenable groups, one could adapt the stable uniqueness theorems of Dadarlat-Eilers [45, Section 4] to a theorem of the necessary form. Indeed, this is essentially what is done in [43, Theorem 1.5], which is closely related to the special case of Theorem 1.14 where all possible obstructions vanish.

However, this will not suffice for us as our key examples (such as those in Theorems 1.8, 1.9, 1.10, and 1.11) involve non-amenable groups. The existing techniques do not seem to work in this case as the maximal group C^*-algebra $C^*(\Gamma)$ will not even be *exact*[12], a technical assumption needed to establish the relevant theorems: compare for example [45, Theorem 4.15] and [43, Theorem 5.4]. We thus give a different approach to stable uniqueness theorems based on a new model of K-homology [139] established by the author and Yu; this has the advantage that it allows one to work directly with ucp maps. This new stable uniqueness theorem is probably the most technically novel aspect of the current paper: see Theorem 4.2 below

Before stating the main stable uniqueness result in the context of group C^*-algebras, we need two definitions.

Definition 1.12. A C^*-algebra is *residually finite dimensional* (RFD) if finite dimensional representations separate points. We say a group Γ is RFD if its maximal group C^*-algebra is RFD.

[11] "Stable" is here used in the K-theoretic sense of 'up to block sum with something'; it is not directly connected to a group being stable as in Definition 1.4.

[12] Exactness of $C^*(\Gamma)$ is a stronger property than the more widely studied exactness of the 'reduced' C^*-algebra $C^*_r(\Gamma)$ and should not be confused with this. Indeed, exactness of $C^*_r(\Gamma)$ is equivalent to Yu's *property A* (see [144, Definition 2.1]) for Γ by the main result of [110], while for residually finite Γ, exactness of $C^*(\Gamma)$ is equivalent to amenability for Γ by [85, Theorem 7.5] (see also [28, Proposition 3.7.11]).

If a finitely generated group is RFD, then it is residually finite; the converse holds if the group is (finitely generated and) amenable, but in general being RFD is significantly stronger than being residually finite. See Remark 5.7 below for more details.

The next definition requires Kasparov's bivariant KK-theory [80] to make sense of. The non-expert reader is encouraged to treat this as a black box.

Definition 1.13. A separable C^*-algebra satisfies the *universal coefficient theorem* (UCT) of Rosenberg and Schochet [123] if it is isomorphic in the bivariant K-theory category KK to a commutative C^*-algebra. We say a group Γ is UCT if its maximal group C^*-algebra satisfies this property.

The UCT for a C^*-algebra A is (very roughly) equivalent to bivariant K-theory $KK(A, \cdot)$ being determined by homomorphisms between the K-theory groups of A and those of other C^*-algebras; this is how we will use it. The UCT holds for all a-T-menable groups as a consequence of a deep theorem of Higson and Kasparov [73] on the Baum-Connes conjecture; this covers amenable groups and also many of the non-amenable groups appearing in Theorem 1.8 through 1.11. We note that we only need an a priori weaker version of the UCT for our main results: see Definition 3.22 and Remark 5.6 below for more on this.

Here is a simplified version of the main theorem where we make an additional assumption on the odd-dimensional K-theory group $K_1(C^*(\Gamma))$ to obviate the need for K-theory with finite coefficients: see Theorem 5.3 below for the full version.

Theorem 1.14. *Let Γ be a discrete group with a fixed finite generating set S. Assume that Γ is RFD, UCT, and also that $K_1(C^*(\Gamma))$ is torsion-free. Then for any $\epsilon > 0$ there exist $\delta > 0$ and a finite subset P of $K_0(C^*(\Gamma))$ with the following property.*

Let $\phi : \Gamma \to M_n(\mathbb{C})$ be a ucp (S, δ)-representation in the sense of Definition 1.3. Then ϕ induces a well-defined 'partial homomorphism' from P to $K_0(\mathbb{C})$. Moreover, if $\pi : \Gamma \to M_n(\mathbb{C})$ is a finite-dimensional unitary representation such that the induced map $\pi_* : K_0(C^*(\Gamma)) \to K_0(\mathbb{C})$ agrees with ϕ_* on P, then there is a finite dimensional unitary representation $\theta : \Gamma \to M_k(\mathbb{C})$ and a unitary $u \in M_{n+k}(\mathbb{C})$ such that

$$\|u(\phi(s) \oplus \theta(s))u^* - (\pi(s) \oplus \theta(s))\| < \epsilon$$

for all $s \in S$.

Complementing the RFD and UCT properties from Definitions 1.12 and 1.13, there is a third important acronym for us: the *local lifting property* (LLP) of Kirchberg [85, Section 2]. If $C^*(\Gamma)$ satisfies the LLP, then one can drop the condition on ϕ being ucp in Theorem 1.14 as in the following theorem.

Theorem 1.15. *Let Γ be a countable discrete group. Assume that Γ is RFD, UCT, LLP, and also that $K_1(C^*(\Gamma))$ is torsion-free. Then for any finite subset S of Γ and any $\epsilon > 0$ there exist a finite subset T of Γ, $\delta > 0$, and a finite subset P of $K_0(C^*(\Gamma))$ with the following property.*

Let $\phi : \Gamma \to M_n(\mathbb{C})$ be a (T, δ)-representation in the sense of Definition 1.3. Then ϕ induces a well-defined 'partial homomorphism' from P to $K_0(\mathbb{C})$. Moreover, if $\pi : \Gamma \to M_n(\mathbb{C})$ is a finite-dimensional unitary representation such that the induced map $\pi_* : K_0(C^*(\Gamma)) \to K_0(\mathbb{C})$ agrees with ϕ_* on P, then there is a finite dimensional unitary representation $\theta : \Gamma \to M_k(\mathbb{C})$ and a unitary $u \in M_{n+k}(\mathbb{C})$ such that

$$\|u(\phi(s) \oplus \theta(s))u^* - (\pi(s) \oplus \theta(s))\| < \epsilon$$

for all $s \in S$.

For an amenable group Γ, the LLP holds for $C^*(\Gamma)$ by the Choi-Effros lifting theorem [35, Theorem 3.10] (see also the simpler proof given in [5, Theorem 7]) for separable nuclear C^*-algebras, and Lance's theorem that $C^*(\Gamma)$ is nuclear for amenable Γ [92, Theorem 4.2]. The LLP holds more generally than for amenable Γ: see Remark 5.5 below for more examples. However, the LLP does not seem to have received much attention from group theorists and relatively little is known here: for example, it is open if surface groups or fundamental groups of hyperbolic 3-manifolds satisfy the LLP, which is the reason "ucp" appears in the conclusions of Theorems 1.8, 1.10, and 1.11 above; it could be dropped if these groups were known to satisfy the LLP and we conjecture that this is indeed the case.

1.5 The Baum-Connes conjecture and index theory

In order to deduce Theorems 1.8, 1.9,1.10, and 1.11 from Theorems 1.14 and 1.15, we need to show that for any quasirepresentation $\phi : \Gamma \to M_n(\mathbb{C})$ and finite subset P of $K_0(C^*(\Gamma))$ satisfying appropriate conditions, there exists an honest representation $\pi : \Gamma \to M_n(\mathbb{C})$ such that the map $\pi_* : K_0(C^*(\Gamma)) \to K_0(\mathbb{C})$ induced by π agrees with the partial homomorphism $\phi_* : P \to K_0(\mathbb{C})$ induced by ϕ.

Assuming that Γ is torsion free, the key ingredient for this is the *Baum-Connes-Kasparov assembly map* [80, 16]

$$\mu : RK_*(B\Gamma) \to K_*(C^*(\Gamma)), \tag{3}$$

which relates the topologically defined K-homology group $RK_*(B\Gamma)$ with the K-theory of group C^*-algebra; see [3] for a survey on this. Following Baum-Douglas [17], the left hand side of line (3) above consists very roughly of continuous maps from closed manifolds (plus some orientation and bundle data) to the classifying space $B\Gamma$ modulo appropriate equivalence relations. If μ is an isomorphism, this in principle allows one to compute the (partially defined) map $\pi_* : K_0(C^*(\Gamma)) \to K_0(\mathbb{C})$ in terms of index theory pairings between Dirac operators on the manifolds making up cycles for $RK_*(B\Gamma)$ with the flat bundle associated to the representation.

Much of this analysis has been done in a beautiful series of papers by Dadarlat (see for example [41, 43, 44]), although what we need to understand the maps $\pi_* : K_0(C^*(\Gamma)) \to K_0(\mathbb{C})$ is relatively simple and essentially part of the folklore of the subject.

The main new index-theoretic observations we make concern finite coefficients. Indeed, we have mentioned above that one should be working not just with the K-theory group $K_0(C^*(\Gamma))$ with integer coefficients, but also with K-theory with finite coefficients $K_0(C^*(\Gamma); \mathbb{Z}/n)$, as introduced by Schochet [124]. The assumption in Theorem 1.14 that $K_1(C^*(\Gamma))$ is torsion free is made precisely as it allows us to avoid this issue.

In general, however, given a quasi-representation $\phi : \Gamma \to M_n(\mathbb{C})$ it also induces partially defined maps ϕ_* on K-theory with finite coefficients, and we need to find a representation $\pi : \Gamma \to M_n(\mathbb{C})$ such that $\pi_* : K_0(C^*(\Gamma); \mathbb{Z}/n) \to K_0(\mathbb{C}; \mathbb{Z}/n)$ agrees with ϕ_* (where the latter is defined) for all $n \geqslant 2$. We do this computation using the *relative eta invariants* of Atiyah, Patodi, and Singer [12]. Very roughly, relative eta invariants measure how the spectrum of a differential operator shifts when it is 'twisted' by a finite-dimensional representation. The key technical results are Theorem 6.10 which relates the map on K-theory with finite coefficients induced by a representation to relative eta invariants; and Theorem 6.15 which uses this to show that the map induced on K-theory, including K-theory with finite coefficients, can be 'matched' by the map induced by an honest representation, under appropriate assumptions.

After our new stable uniqueness theorem, this application of relative eta invariants to representation theory is the second main technical novelty in the current paper. We suspect, however, that some of the computations we need here may be know to experts: in particular Theorem 6.10 is in some sense implicit in the original three papers of Atiyah, Patodi, and Singer [11, 12, 13].

1.6 Further questions

Here we summarize (a small selection of the!) issues that we do not even start to address in this paper; we mention these mainly to highlight problems that we consider interesting for future work.

One of the most interesting applications of representation stability is in obstructing approximation results for groups. The paradigm here, following [34], proceeds roughly as follows. Assume one wants to show that a particular group Γ does not admit a separating family of suitably good quasi-representations. Roughly, this follows if Γ is stable, and also if it does not admit a separating family of honest representations. Our methods *require* that Γ has a separating family of honest representations, and so are (a priori) useless for results along these lines.

A second limitation to our methods is the reliance on low-dimensionality assumptions (on classifying spaces) for concrete examples. There seems to be a lot of potential for higher-dimensional computations, but this would require new ideas.

Another issue we have nothing to say about is what happens in the presence of finite subgroups. There is probably quite a lot that can be said here, perhaps in terms of Lott's *delocalized invariants* [101] or related machinery. We did not attempt to address this here, but this is only due to the author's limited knowledge and energy, not because we are aware of serious obstructions. There is a partial obstruction in that our methods require finite-dimensional classifying spaces for concrete examples, and the classifying space of a group with torsion is always infinite-dimensional. However, it is plausibly possible to get around this by considering the classifying space for proper actions or something similar. This again seems an interesting avenue for further work.

A final issue that seems quite mysterious (and that this paper has nothing new to say about) is the extent of validity of our RFD, LLP, and UCT assumptions. The status is perhaps worst for the LLP, where very little seems to be known for interesting groups of geometric or combinatorial origin like surface groups and other one-relator groups. As already commented, RFD, LLP, and UCT are all known for amenable groups (one needs to also assume MAP for RFD), and all are known to hold at least a little more generally than this: see Remarks 5.5, 5.6, and 5.7 below. Moreover it seems (some version of) property (T) obstructs (some version of) each of these properties: see for example [21] for the RFD property, [76] for the LLP, and [128] for the UCT. However, there are no really general obstruction results along these lines that we are aware of. It is also worth pointing out that the versions of the LLP and UCT we need are a priori weaker than the full versions (see Definitions 2.4 and 3.22), but to what extent this is 'really' true is quite unclear to us. It would be very interesting to see further progress along these lines.

1.7 Outline of the paper

Section 2 discusses quasi-representations in more detail: in particular, we clarify the meaning of the ucp assumption, and the role of Kirchberg's LLP in allowing the approximation of general quasi-representations by ucp quasi-representations.

Section 3 recalls the controlled K-homology groups from [139], and relates them to homomorphisms between K-theory groups using the universal multicoefficient theorem of Dadarlat and Loring [47] and Dadarlat's work on the topology of KK-theory [40]. This is then used to establish our C^*-algebraic stable uniqueness theorem in Section 4. Sections 3 and 4 are written in the language of abstract C^*-algebras; in Section 5 we specialize to group C^*-algebras and state our main result for conditional weak stability of ucp quasi-representations. Section 5 also surveys when the various assumptions (RFD, UCT, LLP) going into the main theorem are known for some concrete classes of groups: all three assumptions are known for amenable groups and somewhat beyond this class, but the state of knowledge is only partial, and there is strong evidence all are obstructed by appropriate forms of property (T).

Section 6 brings the Baum-Connes conjecture and index-theoretic methods into play. The main technical result is Theorem 6.15 which lets us match the K-theoretic data associated to a quasi-representation with K-theoretic data from an honest representation. Finally, Section 7 specializes to low-dimensional (meaning the classifying space is low-dimensional) examples where explicit computations are possible: in particular, it establishes Theorems 1.8, 1.9, 1.10, and 1.11 above.

1.8 Notation and conventions

We write $K_*(A) := K_0(A) \oplus K_1(A)$ for the $\mathbb{Z}/2$-graded K-theory group of a C^*-algebra A, and similarly $K^*(A) := K^0(A) \oplus K^1(A)$ defines the $\mathbb{Z}/2$-graded K-homology group. For a compact metrizable space X, we write $K^*(X)$ and $K_*(X)$ for K-theory and K-homology; thus $K^*(X) = K_*(C(X))$ and $K_*(X) = K^*(C(X))$ where $C(X)$ is the C^*-algebra of continuous complex-valued functions on X. A homomorphism between $\mathbb{Z}/2$-graded groups is *graded* if it splits as a direct sum of homomorphisms that preserve the components.

We write 1_n and 0_n for the unit and zero element of $M_n(A)$, where A is a unital C^*-algebra. For a C^*-algebra A, $a \in M_n(A)$ and $b \in M_m(A)$, define $a \oplus b := \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \in M_{n+m}(A)$ for the usual block sum matrix.

We assume the groups we work with (essentially, anything called Γ) are countable and discrete. Countability is probably not strictly necessary – we would guess that most arguments could be extended to the general case through taking appropriate limits – but the countable case covers all the examples we are interested in, and doing everything in general would require a lot more pedantry with KK-theory and the UCT (amongst other things) than seemed likely to be helpful to the reader.

A subset S of a group is called *symmetric* if it is closed under taking inverses. A subset X of a C^*-algebra is *symmetric* if it is closed under taking adjoints.

Throughout the paper, $\tau : \Gamma \to \mathbb{C}$ is the trivial representation of a group Γ, and $\tau^{(n)} : \Gamma \to M_n(\mathbb{C})$ is the n-dimensional trivial representation; we will occasionally also write τ for $\tau^{(n)}$ if there does not seem to be a risk of confusion. We also use the same notation for the maps induced on $C^*(\Gamma)$ by these representations.

We will sometimes need to use Kasparov's KK-theory: see [22, Chapter VIII] or [80, Section 2] for background. For any pair of separable C^*-algebras A, B, KK-theory assigns an abelian group $KK(A, B)$ in such a way that $KK(\mathbb{C}, A)$ and $KK(A, \mathbb{C})$ canonically identify with the K-theory $K_0(A)$ and K-homology $K^0(A)$ respectvely. If C, D, E are separable C^*-algebras and "$\otimes$" denotes the spatial tensor product of C^*-algebras or tensor product of abelian groups as appropriate, then there is a pairing

$$KK(A, B \otimes C) \otimes KK(C \otimes D, E) \to KK(A \otimes D, B \otimes E) \tag{4}$$

with strong formal properties including appropriate versions of associativity: see [80, Definition 2.12 and Theorem 2.14] or [22, Section 18.9]. The most important of these pairings for us will be the pairing

$$KK(\mathbb{C}, A) \otimes KK(A, \mathbb{C}) \to KK(\mathbb{C}, \mathbb{C}) = \mathbb{Z},$$

between K-theory and K-homology. Another important special case for us are the pairings

$$KK(\mathbb{C}, A \otimes O_{n+1}) \otimes KK(A, \mathbb{C}) \to KK(\mathbb{C}, O_{n+1}) = \mathbb{Z}/n$$

where O_{n+1} is the Cuntz algebra of [37]. These will be important for us when we discuss K-theory with $\mathbb{Z}/n$ coefficients, as for us $KK(\mathbb{C}, A \otimes O_{n+1})$ equals the K-theory group of A with $\mathbb{Z}/n$ coefficients, denoted $K_0(A; \mathbb{Z}/n)$. Finally, we note that the group $KK(A, B)$ is often denoted $KK_0(A, B)$, and thought of as the *even* KK-theory group; there is also an *odd* group $KK_1(A, B)$, and a pairing

$$KK_i(A, B \otimes C) \otimes KK_j(C \otimes D, E) \to KK_{i+j \bmod 2}(A \otimes D, B \otimes E)$$

defined for any separable C^*-algebras.

1.9 Acknowledgments

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2 Ucp quasi-representations and the local lifting property

In this section we discuss some general considerations about the relationship between different types of quasi-representation. We believe the results here are essentially folklore; we give proofs where we could not find a result in the literature.

The following lemma will not be used directly in the paper: the point of including it is to relate our notion of unit ball-valued quasi-representations from Definition 1.3 with more the more usual unitary-valued version.

Lemma 2.1. *Let Γ be a discrete group, let $S \subseteq \Gamma$ be symmetric, and let $\epsilon \in (0,1)$. Let $\pi : \Gamma \to \mathcal{B}(H)_1$ be an (S,ϵ)-representation as in Definition 1.3. Then there exists an $(S,6\epsilon)$-representation $\sigma : \Gamma \to \mathcal{B}(H)_1$ such that $\|\sigma(s) - \pi(s)\| < \epsilon$ for all $s \in S$ and such that σ takes values in the unitary operators.*

Proof. For each $s \in S$, we have that $\|\pi(s)\pi(s^{-1}) - 1\| < \epsilon$ and $\|\pi(s^{-1})\pi(s) - 1\|$. Let $b = \pi(s^{-1})\pi(s)$. Then b is invertible with

$$b^{-1} = \sum_{n=0}^{\infty} (1-b)^n$$

whence $\|b^{-1}\| < (1-\epsilon)^{-1}$. Hence $\pi(s)$ is invertible with $\pi(s)^{-1} = \pi(s)b^{-1}$, whence also

$$\|\pi(s)^{-1}\| \leqslant \|\pi(s)\|\|b^{-1}\| < (1-\epsilon)^{-1}.$$

Hence

$$1 = \|\pi(s)\pi(s)^{-1}\| < \|\pi(s)\|(1-\epsilon)^{-1},$$

and so $\|\pi(s)\| > 1-\epsilon$. Let now $\pi(s) = u_s a_s$ be the polar decomposition of $\pi(s)$ (see for example [23, pages 21-23]) where u_s is unitary and a_s is positive and invertible. We have $\|a_s\| = \|\pi(s)\| \leqslant 1$, and that $\pi(s)^{-1} = a_s^{-1}u_s^*$ whence $\|a_s^{-1}\| = \|\pi(s)^{-1}\| < (1-\epsilon)^{-1}$. As a_s and a_s^{-1} are positive, their norms are the maximal elements of their spectrum so these inequalities imply that the spectrum of a_s is contained in $(1-\epsilon, 1]$. The functional calculus then implies that $\|a_s - 1\| < \epsilon$. Hence

$$\|\pi(s) - u_s\| < \epsilon \tag{5}$$

for each $s \in S$.

Now, for each $t \in S^2 \backslash (S \cup \{e\})$, choose $l(t)$ and $r(t)$ in S with $t = l(t)r(t)$. Define $\sigma : \Gamma \to \mathcal{U}(H)$

$$\sigma(g) = \begin{cases} u_s & g = s \in S \\ l(t)r(t) & g = t \in S^2 \backslash (S \cup \{e\}) \\ 1 & \text{otherwise} \end{cases}$$

From line (5), we have $\|\pi(s)-\sigma(s)\| < \epsilon$ for all $s \in S$. To check that σ is an $(S, 6\epsilon)$-representation, we need to check that $\|\sigma(s_1)\sigma(s_2) - \sigma(s_1 s_2)\| < \epsilon$ for all $s_1, s_2 \in S$. Computing:

$$\begin{aligned}\|\sigma(s_1)\sigma(s_2) - \sigma(s_1 s_2)\| &\leqslant \|\pi(s_1)\pi(s_2) - \sigma(s_1 s_2)\| + \|(\pi(s_1) - u_{s_1})\pi(s_2)\| \\ &\quad + \|u_{s_1}(\pi(s_2) - u_{s_2})\| \\ &< 2\epsilon + \|\pi(s_1)\pi(s_2) - \sigma(s_1 s_2)\|. \qquad (6)\end{aligned}$$

There are three cases: if $s_1 = s_2^{-1}$, then $\sigma(s_1 s_2) = 1$ and $\|\pi(s_1)\pi(s_2) - 1\| = \|\pi(s_1 s_2) - \pi(s_1)\pi(s_2)\| < \epsilon$, and we are done by line (6); if $s_1 s_2 = s \in S$, then we have

$$\|\pi(s_1)\pi(s_2) - \sigma(s_1 s_2)\| \leqslant \|\pi(s) - \sigma(s) - (\pi(s) - \pi(s_1 s_2))\| < 2\epsilon$$

and again are done by line (6); finally, if $s_1 s_2 = t = l(t)r(t) \in S^2 \backslash (S \cup \{e\})$, then

$$\begin{aligned}\|\pi(s_1)\pi(s_2) - \sigma(s_1 s_2)\| &\leqslant \|\pi(s_1)\pi(s_2) - \pi(t)\| + \|\pi(t) - \pi(l(t))\pi(r(t))\|\| \\ &\quad + \|(\pi(l(t)) - u_{l(t)})\pi(r(t))\| + \|u_{l(t)}(\pi(r(t)) - u_{r(t)})\| \\ &< 4\epsilon\end{aligned}$$

and again we are done by line (6). ☐

One of the main reasons that ucp quasi-representations are useful is the next result, which is classical; the reader might also compare [86, Proposition 2.3], which gives a more detailed result for tracial approximations (and when the group has property (T)). To state it, we recall that a linear map $\phi : A \to \mathcal{B}(H)$ from a $*$-algebra A to $\mathcal{B}(H)$ is *unital completely positive* (ucp) if it is unital, and if for all n and any $a \in M_n(A)$, if $\phi_n : M_n(A) \to M_n(\mathcal{B}(H))$ is the map defined by applying ϕ entrywise, then the element $\phi(a^* a) \in M_n(\mathcal{B}(H))$ is a positive operator.

For more background on ucp maps between C^*-algebras and related concepts we use below like operator systems and the LLP, we recommend for example [28], [116], or [113].

Proposition 2.2. *Let $\phi : \Gamma \to \mathcal{B}(H)_1$ be a ucp quasi-representation in the sense of Definition 1.3.*

(i) There exists another Hilbert space H', an isometry $v : H \to H'$, and a unitary representation $\pi : \Gamma \to \mathcal{B}(H)$ such that $\phi(g) = v^* \pi(g) v$ for all $a \in \mathbb{C}[\Gamma]$.

(ii) There is a unique ucp map $\phi : C^*(\Gamma) \to \mathcal{B}(H)$ extending ϕ by linearity and continuity.

(iii) If S is a symmetric subset of Γ and if $\epsilon > 0$ is such that

$$\|\phi(s)^* \phi(s) - 1\| < \epsilon \quad \text{and} \quad \|\phi(s)\phi(s)^* - 1\| < \epsilon$$

then

$$\|\phi(gs) - \phi(g)\phi(s)\| < \sqrt{\epsilon} \quad \text{and} \quad \|\phi(gs) - \phi(g)\phi(s)\| < \sqrt{\epsilon}$$

for all $g \in \Gamma$. In particular, this holds of ϕ is an (S, ϵ)-representation as in Definition 1.3.

(iv) If ϕ is unitary-valued, then it is a representation.

Proof. Part (i) is essentially due to Naimark [108]; see for example [113, Theorem 4.8] for a modern proof of the precise statement we give. Part (ii) is a direct consequence of part (i): indeed, $\pi : \Gamma \to \mathcal{B}(H')$ extends uniquely to a $*$-homomorphism $\pi : C^*(\Gamma) \to \mathcal{B}(H)$, and one checks directly that the formula $\phi(a) := v^*\pi(a)v$ defines a ucp extension of π to $C^*(\Gamma)$.

For part (iii), let v and π be as in part (i) and define $p := vv^*$, which is a projection on H'. Then for any $s \in S$,

$$\|p\pi(s)^*(1-p)\pi(s)p\| = \|v(v^*v - v\pi(s)^*vv^*\pi(s)v)v^*\| = \|1 - \phi(s)^*\phi(s)\| < \epsilon.$$

Hence by the C^*-identity, $\|(1-p)\pi(s)p\| < \sqrt{\epsilon}$. For any $g \in \Gamma$ and $s \in S$, we therefore have

$$\begin{aligned}\|\phi(gs) - \phi(g)\phi(s)\| &= \|v^*\pi(g)\pi(s)v - v^*\pi(g)vv^*\pi(s)v\| \\ &= \|v^*\pi(g)(1-p)\pi(s)v\|.\end{aligned}$$

As $pv = v$, the right hand side above is at most $\sqrt{\epsilon}$. The inequality $\|\phi(gs) - \phi(g)\phi(s)\| < \sqrt{\epsilon}$ follows similarly on reversing the roles of $\phi(s)$ and $\phi(s)^*$.

Finally, part (iv) follows from part (iii) on taking $S = \Gamma$. □

Remark 2.3. Analogously to part (iii) of Proposition 2.2, if $\phi : A \to B$ is a general ucp map between C^*-algebras and X is a subset of A such that $\|\phi(aa^*) - \phi(a)\phi(a)^*\| < \epsilon$ and $\|\phi(a^*a) - \phi(a)^*\phi(a)\| < \epsilon$ for all $a \in X$, then

$$\|\phi(ab) - \phi(a)\phi(b)\| < \sqrt{\epsilon}\|b\| \quad \text{and} \quad \|\phi(ba) - \phi(b)\phi(a)\| < \sqrt{\epsilon}\|b\|$$

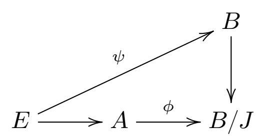
for all $b \in A$ and $a \in X$. The proof is essentially the same.

From Proposition 2.2, we see that the ucp condition is a strong one to put on quasi-representations. Remarkably, however, under an assumption on $C^*(\Gamma)$ that holds quite widely, any quasi-representation can be approximated by a ucp quasi-representation. The goal of the rest of this section is to establish this.

We next give the key definition, which is due to Kirchberg [85, Section 2]. For the statement, recall that an *operator system* is a norm-closed and adjoint-closed subspace of a C^*-algebra that contains the unit. Recall also if $(n_m)_{m=1}^{\infty}$ is a sequence of natural numbers that $\prod_{m=1}^{\infty} M_{n_m}(\mathbb{C})$

denotes the C^*-algebras of bounded sequences (a_m) with $a_m \in M_{n_m}(\mathbb{C})$, and $\bigoplus_{m=1}^{\infty} M_{n_m}(\mathbb{C})$ denotes the ideal in this consisting of sequences (a_m) such that $\|a_m\| \to 0$ as $m \to \infty$.

Definition 2.4. A unital C^*-algebra A has the *local lifting property* (LLP) if for any ucp map $\phi : A \to B/J$ into a quotient C^*-algebra and any finite-dimensional operator system $E \subseteq A$, there exists a ucp map $\psi : E \to B$ that lifts ϕ, i.e. so that the following diagram commutes



(here the unlabeled maps are the canonical inclusion and quotient).

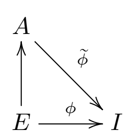
The C^*-algebra has the *weak matricial LLP* if the above conclusion holds in the special case that $B = \prod_{m=1}^{\infty} M_{n_m}(\mathbb{C})$ for some sequence $(n_m)_{m=1}^{\infty}$ of natural numbers, J is the ideal $\bigoplus_{m=1}^{\infty} M_{n_m}(\mathbb{C})$, and ϕ is a unital $*$-homomorphism.

A group Γ has the *(weak matricial) LLP* if its maximal group C^*-algebra $C^*(\Gamma)$ has the (weak matricial) LLP.

See Remark 5.5 below for a brief survey of the class of groups that are known to have the LLP.

We next need to recall the notation of an injective C^*-algebra: see for example [23, Section IV.2.1] for more background.

Definition 2.5. A unital C^*-algebra I is *injective* if for any operator system E in a unital C^*-algebra A and any ucp map $\phi : E \to I$, there exists a ucp extension $\widetilde{\phi} : A \to I$, i.e. so that the diagram



commutes.

The only example of injective C^*-algebras we will need are products of the form $\prod_{m=1}^{\infty} M_{n_m}(\mathbb{C})$: injectivity of such C^*-algebras is a direct consequence of the finite-dimensional case of Arveson's extension theorem as in [4, Theorem 1.2.3] (see for example [113, Theorem 6.2] or [28, Corollary 1.5.16] for textbook treatments).

Variants of the next result are well-known: we got the idea from [98, Lemma 2.1]; [76, Corollary 1.7] is also related. We could not find exactly what we need in the literature, so provide a proof for the reader's convenience.

Proposition 2.6. *Let Γ be a countable group with the weak matricial LLP. Then for any finite symmetric subset S of Γ and any $\epsilon > 0$ there exists a finite subset T of Γ and $\delta > 0$ such that if $\phi : \Gamma \to M_n(\mathbb{C})$ is a (T, δ)-representation in the sense of Definition 1.3 then there exists a ucp (S, ϵ)-representation $\psi : \Gamma \to M_n(\mathbb{C})$ such that $\|\psi(s) - \phi(s)\| < \epsilon$ for all $s \in S$.*

Proof. Assume for contradiction that the statement fails. Let $T_1 \subseteq T_2 \subseteq T_3 \subseteq \cdots$ be a nested collection of finite symmetric subsets of Γ with union all of Γ. Then there exists a finite symmetric subset S of Γ and $\epsilon > 0$ such that for any $m \geqslant 1$ there are $n_m \in \mathbb{N}$ and a $(T_m, 1/m)$-representation $\phi_m : \Gamma \to M_{n_m}(\mathbb{C})$ such that for any ucp (S, ϵ)-representation $\psi : \Gamma \to M_{n_m}(\mathbb{C})$, we have $\|\psi(s) - \phi_m(s)\| \geqslant \epsilon$ for all $s \in S$

Define M to be the C^*-algebra $\prod_{m=1}^{\infty} M_{n_m}$, and let M_0 be the ideal $\bigoplus_{m=1}^{\infty} M_{n_m}(\mathbb{C})$. Define $M_\infty := M/M_0$ and define

$$\Phi : \Gamma \to M_\infty, \quad g \mapsto [\phi_1(g), \phi_2(g), \cdots].$$

This is a homomorphism into the unitary group of M_∞, so extends uniquely by linearity and continuity to a unital $*$-homomorphism $\Phi : C^*(\Gamma) \to M_\infty$. Let $E \subseteq C^*(\Gamma)$ be the (finite-dimensional) operator system spanned by $\{1\} \cup S \cup S^2$. Then by the weak matricial LLP for $C^*(\Gamma)$ there is a ucp map $\Psi : E \to M$ such that Ψ lifts $\Phi|_E$. As M is an injective C^*-algebra, we may extend Ψ to a ucp map $\Psi : C^*(\Gamma) \to M$. Write $\psi_m : C^*(\Gamma) \to M_{n_m}$ for the compositions of Ψ with the canonical quotient map $M \to M_{n_m}$ defined by evaluating at the m^{th} coordinate, and note that ψ_m is also ucp.

Note now that the sequence $(\phi_m(s))_{m=1}^{\infty} \in M$ is bounded by definition of a quasi-representation and in particular, it is an element of M that lifts $\Phi(s)$. As $\Psi(s)$ also lifts $\Phi(s)$, we have that $\Psi(s) - (\phi_m(s)) \in M_0$, i.e. that $\psi_m(s) - \phi_m(s) \to 0$ as $m \to \infty$. It follows that for suitably large m, ψ_m is a ucp (S, ϵ)-representation, and satisfies $\|\psi_m(s) - \phi_m(s)\| < \epsilon$ for all $s \in S$, contradicting the assumption. ☐

Let us make a few comments on the relation of the LLP and weak-matricial LLP.

Remark 2.7. The version of the LLP where one stipulates that *ucp* maps from A into B/J where $B = \prod_{m=1}^{\infty} M_{n_m}(\mathbb{C})$ and $J = \oplus_{m=1}^{\infty} M_{n_m}(\mathbb{C})$ lift to ucp maps is equivalent to the general case of the LLP: this follows from arguments of Ozawa in [111]; see particularly [111, Remark 2.11].

On the other hand, the weak matricial LLP is genuinely weaker than the usual LLP. Indeed, there are C^*-algebras without the LLP that do not admit any non-zero $*$-homomorphisms into B/J as above. For example, any C^*-algebra admitting such a $*$-homomorphism has a non-trivial stably finite quotient and this is not the case for $\mathcal{B}(\ell^2(\mathbb{N}))$; this C^*-algebra is known not to have the LLP as a consequence of [78, Corollary 3.1] (see also [28, Corollary 13.2.5 and Theorem 13.5.1] for a textbook exposition).

We do not know if the weak matricial LLP is genuinely weaker than the LLP for group C^*-algebras, or even for RFD group C^*-algebras. It was pointed out to us by Tatiana Shulman that the weak matricial LLP for A is equivalent to the Brown-Douglas-Fillmore semigroup $Ext(A)$ (see for example [74, Chapter 2]) being a group whenever A is an RFD C^*-algebra, and that it is open whether $Ext(A)$ being a group is equivalent to the LLP in this level of generality.

3 Controlled K-homology, KL-theory, and total K-theory

Our goal in this section is to recall the definition of controlled K-homology and its relation with KL-theory from [139]. We then relate that to the action of KL-theory on total K-theory from [47, 40]. Throughout this section, anything called A is a separable unital C^*-algebra; we are only interested in the case that $A = C^*(\Gamma)$, but the extra generality makes no difference.

This section is perhaps the most technical of the paper, and we allow ourselves to skip some standard K-theoretic details where this seemed unlikely to cause confusion. We hope a background in C^*-algebra K-theory at the level of [122] should be enough to understand this material: although we use K-homology, KL-homology, and a very small amount of KK-theory, we only really need formal properties of these other theories.

3.1 Controlled K-homology

We start with the definition of controlled K-homology from [139]. Let us say that a representation of a C^*-algebra is *infinitely amplified* if it is a countably infinite direct sum of some other representation.

Definition 3.1. Let A be a separable unital C^*-algebra. Fix a faithful, unital, infinitely amplified representation $A \subseteq \mathcal{B}(H)$ of A on a separable Hilbert space H.

For $\epsilon > 0$ and a finite subset X of the unit ball of A, define

$$\mathcal{P}_\epsilon(X) := \left\{ p \in M_2(\mathcal{B}(H)) \;\middle|\; \begin{array}{l} p = p^* = p^2,\ p - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(\mathcal{K}(H)), \\ \text{and } \|[p,x]\| < \epsilon \text{ for all } x \in X \end{array} \right\}.$$

Define the *controlled (even) K-homology group*

$$K^0_\epsilon(X) := \pi_0(\mathcal{P}_\epsilon(X))$$

to be the set of path components of $\mathcal{P}_\epsilon(X)$. The group operation on $K_\epsilon^0(X)$ is defined by

$$[p] + [q] := [sps^* + tqt^*]$$

where s and t are isometries in the commutant of A satisfying the Cuntz O_2-relation $ss^* + tt^* = 1$ (the choice of such isometries does not matter: we refer to [139, Proposition 6.5] for details, but will not need to use the specifics).

With this structure, each $K_\epsilon^0(X)$ is a countable abelian group. It does not depend on the choice of representation by (a slight variant of) [139, Lemma 6.6].

Definition 3.2. Let $\phi : A \to B$ be a linear map from a $*$-algebra to a C^*-algebra, let X be a finite subset of A, and let $\epsilon > 0$. Then ϕ is called an (X, ϵ)-$*$-*homomorphism* if

$$\|\phi(x^*) - \phi(x)^*\| < \epsilon^{13} \quad \text{and} \quad \|\phi(xy) - \phi(x)\phi(y)\| < \epsilon.$$

for all $x, y \in X$.

Example 3.3. The most important examples of (X, ϵ)-$*$-homomorphisms for us come from almost commuting projections. Indeed, let X be a finite subset of the unit ball of a C^*-algebra A, $\pi : A \to B$ be a $*$-homomorphism. Let $\epsilon > 0$ and let $p \in B$ be a projection such that $\|[p, x]\| < \epsilon$ for all $x \in X$. Then one checks directly that the map

$$\phi : A \to B, \quad a \mapsto p\pi(a)p$$

is an (X, ϵ)-$*$-homomorphism. A map of the form above is moreover ucp, and in fact any ucp (X, ϵ)-$*$-homomorphism arises like this (at the price of making ϵ a little bigger): compare the proof of Lemma 4.4 below.

Definition 3.4. Let A be a separable, unital C^*-algebra. Let $\mathcal{I}_A$ be the set of ordered pairs (X, ϵ) where X is a finite subset of the unit ball of A, and $\epsilon > 0$. We order $\mathcal{I}_A$ by stipulating that $(X, \epsilon) \leqslant (Y, \delta)$ if whenever $\phi : A \to B$ is a ucp (Y, δ)-$*$-homomorphism in the sense of Definition 3.2, then ϕ is also an (X, ϵ)-$*$-homomorphism.

To settle conventions, it will be convenient to recall a standard definition from algebra.

Definition 3.5. Let I be a directed set, and let $(G_i)_{i \in I}$ be an inverse system of abelian groups, i.e. for each $j \geqslant i$, there are homomorphisms $\phi_{ij} : G_j \to G_i$ such that $\phi_{ij} \circ \phi_{jk} = \phi_{ik}$ for all $k \geqslant j \geqslant i$, and such that ϕ_{ii} is the identity on G_i for all $i \in I$.

[13] We will typically apply this definition to ucp maps, in which case "$\phi(x^*) = \phi(x)^*$" holds automatically. For $*$-preserving linear maps, what we call "being an (X, ϵ)-$*$-homomorphism" is often referred to as being "(X, ϵ)-multiplicative" in the literature.

The *inverse limit* of an inverse system $(G_i)_{i\in I}$ with connecting maps $(\phi_{ij})_{i,j\in\mathcal{I},j\geqslant i}$ is defined to be

$$\varprojlim G_i := \left\{(g_i) \in \prod_{i\in I} G_i \mid \phi_{ij}(g_j) = g_i \text{ for all } j \geqslant i\right\}.$$

If each G_i is a topological[14] abelian group then $\varprojlim G_i$ is equipped with the subspace topology it inherits from the product topology on $\prod G_i$.

Remark 3.6. Let $\mathcal{I}_A$ be as in Definition 3.4.

(i) The set $\mathcal{I}_A$ is directed: an upper bound for (X,ϵ) and (Y,δ) is $(X \cup Y, \min\{\epsilon,\delta\})$.

(ii) If $(X,\epsilon) \leqslant (Y,\delta)$ then we have an inclusion

$$\mathcal{P}_\delta(Y) \subseteq \mathcal{P}_\epsilon(X)$$

(consider the ucp map defined by compression by an element of $\mathcal{P}_\delta(Y)$ as in Example 3.3). Thus there is a canonical 'forgetful map'

$$\theta : K^0_\delta(Y) \to K^0_\epsilon(X)$$

induced by the identity inclusion $\mathcal{P}_\delta(Y) \to \mathcal{P}_\epsilon(X)$. These forgetful maps make the family of groups $(K^0_\epsilon(X))_{(X,\epsilon)\in\mathcal{I}_A}$ into an inverse system.

(iii) Recall that a subset S of a directed set I is *cofinal* if for all $i \in I$ there is $s \in S$ with $s \geqslant i$. The directed set $\mathcal{I}_A$ has cofinal sequences: for example, if $X_1 \subseteq X_2 \subseteq \cdots$ is a nested sequence of subsets of the unit ball of A with dense union and $\epsilon_n \searrow 0$, then $((X_n,\epsilon_n))_{n=1}^\infty$ is cofinal. Moreover, if A is finitely generated by some subset X of its unit ball, then any sequence $((X,\epsilon_n))_{n=1}^\infty$ with $\epsilon_n \searrow 0$ is cofinal.

Passing to a cofinal subset does not affect an inverse limit. As such, for our applications, we could replace $\mathcal{I}_A$ by a sequence; this is occasionally technically useful, but we will not do so in this paper.

Definition 3.7. We write $\varprojlim K^0_\epsilon(X)$ for the inverse limit of the inverse system $(K^0_\epsilon(X))_{(X,\epsilon)\in\mathcal{I}_A}$ from Remark 3.6 part (ii), equipped with the topology it inherits by equipping each $K^\epsilon_0(X)$ with the discrete topology.

We now recall the relation of the group from Definition 3.7 to analytic K-homology. The latter was introduced by Kasparov [79] and Brown-Douglas-Fillmore [27]; see [74] for a textbook treatment. The (even) K-homology group $K^0(A)$ of a separable C^*-algebra is an abelian group equipped with a canonical (possibly non-Hausdorff) topology: see [27, 8.5] for the original version of this topology, and [40] for a definitive modern

[14]Typically discrete in our applications.

treatment (in a more general context). Let $\overline{\{0\}}$ denote the closure of 0 in $K^0(A)$, and following Dadarlat [40, Section 5][15] define

$$KL^0(A) := K^0(A)/\overline{\{0\}} \tag{7}$$

to be the associated maximal Hausdorff quotient group. We call $KL^0(A)$ the KL-*homology group* of A.

The following result is (a special case of part of) [139, Proposition A.18].

Theorem 3.8. *For any unital separable C^*-algebra A, there is a canonical homeomorphism*

$$\Theta : KL^0(A) \to \lim_{\leftarrow} K^0_\epsilon(X). \qquad \square$$

3.2 Controlled K-theory with finite coefficients

In this subsection, we recall the definition of K-theory with finite coefficients and introduce a controlled variant of this. We then recall the coefficient change operations and use these to bundle all the K-theory groups with various coefficients into a single object and discuss homomorphisms between these objects. Finally, we relate the group $\lim_{\leftarrow} K^0_\epsilon(X)$ of Definition 3.7 above to this object.

K-theory with finite coefficients was introduced into C^*-algebra theory by Schochet [124], based on a construction of Dold for generalized homology theories [52, Section 1]. The original definition (see [124, Definition 1.2]) is $K_*(A;\mathbb{Z}/n) := K_*(A \otimes C_n)$ for a particular commutative, non-unital C^*-algebra C_n that satisfies

$$K_0(C_n) = \mathbb{Z}/n \quad \text{and} \quad K_1(C_n) = 0. \tag{8}$$

One can replace C_n with any 'reasonable' C^*-algebra with the K-theory groups as in line (8) above: see [124, Theorem 6.4]. For us, it will be convenient to use the Cuntz algebras O_{n+1} for $2 \leqslant n < \infty$, primarily as they are unital[16]. See [37] for the original definition of Cuntz algebras, and [121, Section 4.2] for further background.

Definition 3.9. Let A be a C^*-algebra and let $n \geqslant 2$. The *K-theory of A with coefficients in $\mathbb{Z}/n$* is defined to be

$$K_*(A;\mathbb{Z}/n) := K_*(A \otimes O_{n+1}).$$

[15] The KL groups were originally introduced by Rørdam under more stringent conditions on A using algebraic ideas: see [120, Section 5]. The equivalence of the definition in line (7) with Rørdam's (under Rørdam's conditions on A) follows from [125, Theorem 3.3] or [39, Corollary 4.6].

[16] There is no unital commutative C^*-algebra satisfying the conditions in line (8), so we cannot get away with something commutative.

We will next introduce a 'controlled' variant of K-theory, possibly with finite coefficients. The original version of controlled K-theory is from [143], and is adapted to geometric motivations; the version we give below is adapted to abstract C^*-algebras and ucp (X,ϵ)-$*$-homomorphisms in the sense of Definition 3.2.

Definition 3.10. Let A and B be C^*-algebras with A unital, let X be a finite subset of A, and let $\epsilon > 0$. A projection $p \in A \otimes B$ is (X,ϵ)-*compatible* if whenever $\phi : A \to C$ is a ucp (X,ϵ)-$*$-homomorphism to another C^*-algebra C in the sense of Definition 3.2, we have that[17]

$$\text{spectrum}\big((\phi \otimes \text{id}_B)(p)\big) \subseteq [0,1/4) \cup (3/4,1].$$

Write $\text{Proj}^{X,\epsilon}(A \otimes B)$ for the set of (X,ϵ)-compatible projections in $A \otimes B$.

Definition 3.11. Let A be a unital C^*-algebra, let X be a finite subset of the unit ball of A such that $X = X^*$, and let $\epsilon > 0$. For each $n \geqslant 2$ define

$$\mathcal{Q}^\epsilon(X;\mathbb{Z}/n) := \bigsqcup_{k=1}^{\infty} \text{Proj}^{\epsilon,X}(A \otimes M_k(\mathbb{C}) \otimes O_{n+1}).$$

Let $\sim$ be the equivalence relation on $\mathcal{Q}^\epsilon(X;\mathbb{Z}/n)$ generated by the conditions below:

(i) $p \sim q$ if both are in the same path component of some $\text{Proj}^{\epsilon,X}(A \otimes M_k(\mathbb{C}) \otimes O_{n+1})$;

(ii) $p \sim q$ if $p \in \text{Proj}^{\epsilon,X}(A \otimes M_k(\mathbb{C}) \otimes O_{n+1})$ and there is $m \geqslant 1$ such that $q \in \text{Proj}^{\epsilon,X}(A \otimes M_{k+m}(\mathbb{C}) \otimes O_{n+1})$ and $q = p \oplus 0_m$.

Define $\mathcal{V}^\epsilon(X;\mathbb{Z}/n)$ to be $\mathcal{Q}^\epsilon(X;\mathbb{Z}/n)/\sim$, and define $K_0^\epsilon(X;\mathbb{Z}/n)$ to be the Grothendieck group of $\mathcal{V}^\epsilon(X;\mathbb{Z}/n)$.

Define $\mathcal{Q}^\epsilon(X)$, $\mathcal{V}^\epsilon(X)$, and $K_0^\epsilon(X)$ analogously, but omitting the O_{n+1}.

We call the groups $K_0^\epsilon(X;\,\cdot\,)$ *controlled K-theory groups.*

The definition of $K_0^\epsilon(X;\,\cdot\,)$ is motivated by the following basic lemma, whose proof we leave to the reader.

Lemma 3.12. *Let A be a unital C^*-algebra, let X be a finite subset of the unit ball of A such that $X = X^*$, and let $\epsilon > 0$. Let $\phi : A \to B$ be a ucp (X,ϵ)-$*$-homomorphism as in Definition 3.2. Let χ be the characteristic function of $(1/2,\infty)$.*

For any $n \geqslant 2$ and projection $p \in A \otimes M_k(\mathbb{C}) \otimes O_{n+1}$, define

$$\phi_*(p) := \chi((\phi \otimes id_{M_k(\mathbb{C})\otimes O_{n+1}})(p)) \in B \otimes M_k(\mathbb{C}) \otimes O_{n+1}.$$

Then the assignment $[p] \mapsto [\phi_*(p)]$ uniquely determines homomorphisms

$$\phi_* : K_0^\epsilon(X) \to K_0(B)$$

[17] As ϕ is contractive and positive, the spectrum is automatically contained in $[0,1]$; the condition is thus saying that the spectrum avoids $[1/4,3/4]$.

and also

$$\phi_* : K_0^\epsilon(X; \mathbb{Z}/n) \to K_0(B; \mathbb{Z}/n)$$

for each $n \geqslant 2$. □

Note that if we drop the condition that the projections and homotopies used to define $K_0^\epsilon(X; \cdot)$ above are (X, ϵ)-compatible, then we just get $K_0(A; \cdot)$. Thus Lemma 3.12 says that the groups $K_0^\epsilon(X; \cdot)$ are in some sense 'the part[18] of $K_0(A; \cdot)$ on which ucp (X, ϵ)-$*$-homomorphisms act'.

Having introduced these groups, we introduce natural transformations between them. There are several equivalent ways to do this: see for example [47, Section 1.2] or [48, Section 6]. The discussion in [31, Appendix A] gives a recent overview, and in particular [31, Lemma A.4] gives a proof that all 'reasonable' approaches give the same outcome.

Definition 3.13. For notational convenience, define $B_n = \mathbb{C}$ for $n = 1$, and $B_n = O_{n+1}$ for $n \geqslant 2$.

Let Λ_0 be the (small) category with object set $\{[n] \mid n \geqslant 1\}$, and with morphisms given by $\mathrm{Hom}([n], [m]) = KK(B_n, B_m)$. Composition of morphisms given by the Kasparov product.

A Λ_0-*module* is a functor from the category Λ_0 to the category of abelian groups: more concretely, a Λ_0 module is a sequence $\underline{G} = (G_n)_{n=1}^\infty$ such that elements of $KK(B_n, B_m)$ define morphisms from G_n to G_m, compatibly with the relations between these morphisms in the KK-category.

Given Λ_0-modules $\mathcal{G}$ and $\mathcal{H}$, we define

$$\mathrm{Hom}_{\Lambda_0}(\underline{G}, \underline{H})$$

to be the set of natural transformations from $\underline{G}$ to $\underline{H}$: more concretely, an element of this Hom set is a sequence of group homomorphisms $(\alpha_n : G_n \to H_n)_{n=1}^\infty$ that intertwine the morphisms coming from $KK(B_n, B_m)$. We equip $\mathrm{Hom}_{\Lambda_0}(\underline{G}, \underline{H})$ with the abelian group structure defined by

$$(\alpha_n) + (\beta_n) := (\alpha_n + \beta_n)$$

and with the topology of pointwise convergence, i.e. a net $(\alpha^{(i)})_{i \in I}$ converges to α if and only if for every n, and every $g \in G_n$ there exists $i_{n,x} \in I$ such that for all $i \geqslant i_{n,x}$, $\alpha_n^{(i)}(g) = \alpha_n(g)$.

Example 3.14. Let A be a C^*-algebra, and let $\underline{K}_0(A)$ denote the sequence of abelian groups $(G_n)_{n=1}^\infty$ with $G_1 = K_0(A)$ and $G_n = K_0(A; \mathbb{Z}_n)$. Then $\underline{K}_0(A)$ is a Λ_0-module, with the action defined by Kasparov product. We call $\underline{K}_0(A)$ the *total* K-*theory* of A.

Moreover, a homomorphism $\phi : A \to B$ (or more generally, an element of $KK(A, B)$) induces an element ϕ_* of $\mathrm{Hom}_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(B))$ by associativity of the Kasparov product.

[18] The word 'part' may be misleading: there is a canonical map $K_0^\epsilon(X) \to K_0(A)$, but it is probably not injective in general.

Remark 3.15. We warn the reader that what we call $\mathrm{Hom}_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(B))$ is usually not used in favor of a group $\mathrm{Hom}_\Lambda(\underline{K}_*(A), \underline{K}_*(B))$ that also includes the K_1-groups, and additional ('Bockstein') natural transformations that switch degrees, i.e. that map between between $K_0(A; \cdot)$ and $K_1(A; \cdot)$ and vice versa: see [47, Section 1.4]. In our language, Λ can be described as the category with objects $\{[i, n] \mid i \in \{0, 1\}, n \geqslant 1\}$, and where $\mathrm{Hom}([i, n], [j, m]) = KK_{i+j \bmod 2}(B_n, B_m)$ and then $\mathrm{Hom}_\Lambda(\underline{K}_*(A), \underline{K}_*(B))$ is defined analogously to $\mathrm{Hom}_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(B))$. However, the only example we actually use will be $B = \mathbb{C}$, in which case the natural forgetful map

$$\mathrm{Hom}_\Lambda(\underline{K}_*(A), \underline{K}_*(\mathbb{C})) \to \mathrm{Hom}_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(\mathbb{C}))$$

is an isomorphism; thus we hope our approach should cause no confusion.

One can carry out an analogous construction for the controlled K-theory groups of Definition 3.11.

Lemma 3.16. *Let A be a unital C^*-algebra, let X be a finite subset of the unit ball of A such that $X = X^*$, and let $\epsilon > 0$. Let $\underline{K}_0^\epsilon(X)$ denote the sequence of abelian groups $(G_n)_{n=1}^\infty$ with $G_1 = K_0^\epsilon(X)$ and $G_n = K_0^\epsilon(X; \mathbb{Z}_n)$.*

Then $\underline{K}_0^\epsilon(X)$ can be made into a Λ_0-module in a canonical way. Moreover, if $\phi : A \to B$ is a ucp (X, ϵ)-$*$-homomorphism as in Definition 3.2, then the induced maps ϕ_* of Lemma 3.12 define an element

$$\phi_* \in \mathit{Hom}_{\Lambda_0}(\underline{K}_0^\epsilon(X), \underline{K}_0(B)).$$

Proof. Let B_n be the C^*-algebras from Definition 3.13. It follows from the Kirchberg-Phillips classification theorem [114, 87][19] that any element of $KK(B_n, B_m)$ can be realized by a unital $*$-homomorphism $\phi : B_n \to M_k(B_m)$ for an appropriate $k \in \mathbb{N}$[20] (see for example [64, Theorem A], and moreover that any such homomorphism is unique up to asynmptotically unitary equivalence (see for example [64, Theorem B]). One checks directly that unital $*$-homomorphisms induce maps between controlled K-theory groups; and the uniqueness result guarantees that this is well-defined.

Compatibility of this structure with the maps induced by an (X, ϵ)-$*$-homomorphism follows as if $\phi : A \to B$ and $\psi : C \to D$ are ucp maps between C^*-algebras, then

$$(\mathrm{id}_B \otimes \psi) \circ (\phi \otimes \mathrm{id}_C) = (\phi \otimes \mathrm{id}_D) \circ (\mathrm{id}_A \otimes \psi). \qquad \square$$

Our next task is to relate the Λ_0-module $\underline{K}_0(A)$ of Example 3.14 and $\underline{K}_0^\epsilon(X)$ of Lemma 3.16.

[19] We recall only need this for Cuntz algebras, in which case the essential results were known earlier [120, 58]; however, it is probably simpler to just use the full Kirchberg-Phillips theorem. See [64] for a recent approach that is somewhat simpler than the original one of Kirchberg and Phillips.

[20] Including matrix algebras is necessary to ensure that the units go to the correct place.

Let $\mathcal{I}_A$ be the directed set of Definition 3.4. Then for $(X, \epsilon) \leqslant (Y, \delta)$ in $\mathcal{I}_A$ there are canonical maps

$$K_0^\epsilon(X; \mathbb{Z}/n) \to K_0^\delta(Y; \mathbb{Z}/n)$$

induced by the identity inclusions $\mathcal{Q}^\epsilon(X; \mathbb{Z}/n) \to \mathcal{Q}^\delta(Y; \mathbb{Z}/n)$, and similarly for the case with integer coefficients. These maps fit together into maps of Λ_0-modules, so we get a directed system

$$(\underline{K}_0^\epsilon(X))_{(X,\epsilon) \in \mathcal{I}_A}$$

in the category of Λ_0-modules. For each (X, ϵ) there are moreover canonical homomorphisms

$$K_0^\epsilon(X; \mathbb{Z}/n) \to K_0(A; \mathbb{Z}/n) \tag{9}$$

induced by the identity inclusions of (X, ϵ)-compatible projections into all projections, and similarly for the case with integer coefficients. These again fit together to give maps of Λ_0-modules

$$\underline{K}_0^\epsilon(X) \to \underline{K}_0(A). \tag{10}$$

The following lemma can be compared to the analogous results in other versions of controlled K-theory: see for example [68, Proposition 4.9] or [109, Discussion around Remark 1.18].

Lemma 3.17. *Let A be a unital C^*-algebra. The maps of line (10) induce canonical isomorphisms*

$$\lim_{\rightarrow} K_0^\epsilon(X; \mathbb{Z}/n) \cong K_0(A; \mathbb{Z}/n) \tag{11}$$

and similarly

$$\lim_{\rightarrow} K_0^\epsilon(X) \cong K_0(A). \tag{12}$$

These maps fit together to define an isomorphism of

$$\lim_{\rightarrow} \underline{K}_0^\epsilon(X) \cong \underline{K}_0(A)$$

in the category of Λ_0-modules. Finally, this isomorphism induces a canonical algebraic isomorphism

$$\kappa : \lim_{\leftarrow} Hom_{\Lambda_0}(\underline{K}_0^\epsilon(X), \underline{K}_0(B)) \cong Hom_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(B)). \tag{13}$$

that is moreover a homeomorphism if each of the Hom_{Λ_0} groups involved is equipped with the topology of pointwise convergence, and the left hand side is equipped with the inverse limit topology.

Proof. For the isomorphisms of lines (11) and (12), the essential point is that any finite set of projections in $A \otimes M_k(\mathbb{C}) \otimes O_{n+1}$ is (X, ϵ)-compatible for a large enough element of the set $\mathcal{I}_A$ of Definition 3.4: to see that this, approximate the projections by finite sums of elementary tensors, take X to contain all of the elements of A appearing in such tensors (we may assume all of these to be in the unit ball of A), and take ϵ suitably small; we leave the elementary-but-messy details to the reader. Using compactness of the unit interval $[0,1]$, this also implies that any homotopy of projections is (X,ϵ)-compatible for suitable (X,ϵ). The direct limit isomorphisms of lines (11) and (12) follow readily from these observations: we leave the remaining details to the reader.

We leave the direct checks that these isomorphisms induce an isomorphism in the category of Λ_0-modules. The isomorphism of line (13) is essentially just a restatement of the universal property of direct limits, and the statement that it is a homeomorphism also follows directly from the definitions. ☐

We now come to the action of the controlled K-homology groups of Definition 3.1 on the controlled K-theory groups of Definition 3.10.

Lemma 3.18. *Let A be a unital C^*-algebra, let X be a finite subset of the unit ball of A such that $X = X^*$, and let $\epsilon \in (0, 1/24)$. Let p be an element of the set $\mathcal{P}_\epsilon(X)$ of Definition 3.1, and let q be an element of the set $\mathcal{Q}^\epsilon(X;\mathbb{Z}/n\mathbb{Z})$ of Definition 3.10. In particular, recall that $q \in A \otimes M_k(\mathbb{C}) \otimes O_{n+1}$ and $p \in M_2(\mathbb{C}) \otimes \mathcal{B}(H)$, where $A \subseteq \mathcal{B}(H)$ is a fixed infinitely amplified, faithful, unital representation, and p is of the form $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + b$ for some $b \in M_2(\mathcal{K}(H))$. Finally, let χ be the characteristic function of $(1/2, \infty)$.*

Then the formula

$$\langle p, q\rangle := \chi\Big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + (1_{M_2(\mathbb{C})} \otimes q)(b \otimes 1_{M_k(\mathbb{C})\otimes O_{n+1}})(1_{M_2(\mathbb{C})} \otimes q)\Big). \quad (14)$$

gives a well-defined projection in $M_2(\mathcal{B}(H) \otimes M_k(\mathbb{C}) \otimes O_{n+1})$ whose difference with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is in $M_2(\mathcal{K}(H) \otimes M_k(\mathbb{C}) \otimes O_{n+1})$.

This works completely analogously on omitting O_{n+1}.

Proof. For simplicity of notation, let us write q for $1_{M_2(\mathbb{C})} \otimes q$, b for $b \otimes 1_{M_k(\mathbb{C})\otimes O_{n+1}}$, p for $p \otimes 1_{M_k(\mathbb{C})\otimes O_{n+1}}$, and e for $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. Note first that as q is (X,ϵ)-compatible (see Definition 3.10), we have by Example 3.3 that

$$\text{spectrum}(pqp) \subseteq [0, 1/4) \cup (3/4, 1].$$

Writing $p = e + b$ and using that $[e, q] = 0$, we see that

$$pqp = eq + bqe + eqb + bqb;$$

as $\|[q,p]\| = \|[q,b]\| < \epsilon$, this is within 6ϵ of

$$eq + q(be + eb + b^2)q, \quad (15)$$

whence $eq + q(be + eb + b^2)q$ has spectrum contained in $[0,1/2) \cup (1/2,1]$ as $6\epsilon < 1/4$. On the other hand, as $(e+b)^2 = p^2 = p = (e+b)$, we have that $be + eb + b^2 = b$, and so the element in line (15) equals $eq + qbq$. As $e(1-q)$ is orthogonal to this projection, we see that the spectrum of $e + qbq$ is contained in $[0,1/2) \cup (1/2,1]$; however, up to our notational simplifications, this is the element

$$\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + (1_{M_2(\mathbb{C})} \otimes q)(b \otimes 1_{M_k(\mathbb{C}) \otimes O_{n+1}})(1_{M_2(\mathbb{C})} \otimes q)$$

appearing in the statement of the lemma. Hence χ is continuous on the spectrum of this element, and the functional calculus gives that the element $\chi(e + qbq)$ appearing in the statement is a well-defined projection in $M_2(\mathcal{B}(H) \otimes M_k(\mathbb{C}) \otimes O_{n+1})$. On the other hand, on the spectrum of $e+qbq$, χ is a uniform limit of a sequence $(f_k)_{k=1}^{\infty}$ of polynomials with $f_k(0) = 0$ and $f_k(1) = 1$. For any such polynomial, $f_k(e) = e$, and one computes directly that $f_k(e) - f_k(e+qbq)$ lies in the ideal $M_2(\mathcal{K}(H) \otimes M_k(\mathbb{C}) \otimes O_{n+1})$ for each k. Taking the limit in k completes the proof. □

The next result is the main goal of this subsection. At this point, the proof consists of direct checks that we leave to the reader.

Proposition 3.19. *Let A be a unital C^*-algebra, let X be a finite subset of the unit ball of A such that $X = X^*$, and let $\epsilon \in (0, 1/100)$. Let p be an element of the set $\mathcal{P}_\epsilon(X)$ of Definition 3.1. Then with notation as in Lemma 3.18, the formula*

$$\Phi^{(X,\epsilon)}(p) : [q] \to [\langle p, q\rangle] - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$$

induces a well-defined homomorphism

$$\Phi^{(X,\epsilon)}(p) : K_0^\epsilon(X; \mathbb{Z}/n) \to K_0(\mathcal{K}(H) \otimes O_{n+1}) \cong K_0(\mathbb{C}; \mathbb{Z}/n)$$

and similarly

$$\Phi^{(X,\epsilon)}(p) : K_0^\epsilon(X) \to K_0(\mathcal{K}(H)) \cong K_0(\mathbb{C}).$$

These homomorphisms fit together to give a well-defined homomorphism

$$\Phi^{(X,\epsilon)} : K_0^\epsilon(X) \to Hom_{\Lambda_0}(\underline{K}_0^\epsilon(X), \underline{K}_0(\mathbb{C}))$$

that is continuous when the left hand side is equipped with the discrete topology, and the right hand side has the topology of pointwise convergence.

Finally, taking inverse limits over the set $\mathcal{I}_A$ of Definition 3.4 and applying Lemma 3.17 gives a homomorphism

$$\Phi : \lim_{\leftarrow} K_\epsilon^0(X) \to Hom_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(\mathbb{C})), \tag{16}$$

which is moreover continuous when $Hom_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(\mathbb{C}))$ is equipped with the topology of pointwise convergence. □

We will need one more piece of terminology and observation for later sections.

Definition 3.20. Let A be a C^*-algebra. We define a K-*datum* P for A to be a collection $\{P_n\}_{n\in F}$ where F is a finite subset of $\mathbb{N}$, and each P_n is a finite subset of $K_0(A;\mathbb{Z}/n)$ for $n \geqslant 2$, or of $K_0(A)$ for $n = 1$.

Given a K-datum P and another C^*-algebra B, we write $\mathrm{Hom}_{\Lambda_0}(P, \underline{K}_0(B))$ for the quotient of $\mathrm{Hom}_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(B))$ by the equivalence relation defined by: $\alpha \sim \beta$ if α_n and β_n agree on all elements of P_n for all $n \geqslant 1$.

Lemma 3.21. *Let A be a unital C^*-algebra, and let B be abother C^*-algebra. For any K-datum P there exists an element (X, ϵ) of the directed set $\mathcal{I}_A$ of Definition 3.4 with the following properties:*

(i)

$$Hom_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(B)) \to Hom_{\Lambda_0}(P, \underline{K}_0(B))$$

factors as the composition of the canonical quotient map

$$Hom_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(B)) \to Hom_{\Lambda_0}(\underline{K}_0^{\epsilon}(X), \underline{K}_0(B))$$

arising from the isomorphism κ of line (13), and a homomorphism

$$\kappa_P^{(X,\epsilon)} : Hom_{\Lambda_0}(\underline{K}_0^{\epsilon}(X), \underline{K}_0(\mathbb{C})) \to Hom_{\Lambda_0}(P, \underline{K}_0(\mathbb{C})).$$

(ii) There exists $\gamma > 0$ such that if $\phi, \psi : A \to M_n(\mathbb{C})$ are ucp (X, ϵ)-$*$-homomorphisms such that

$$\|\phi(x) - \psi(x)\| < \gamma$$

for all $x \in X$, then with notation as in Lemma 3.16

$$\kappa_P^{(X,\epsilon)}(\phi_*) = \kappa_P^{(X,\epsilon)}(\psi_*).$$

Proof. Let P be as given. Let $(X, \epsilon) \in \mathcal{I}_A$ be large enough so that P is contained in the image of the canonical map $\underline{K}_0^{\epsilon}(X) \to \underline{K}_0(A)$ as in Lemma 3.17. It is straightforward to check that (i) holds.

For (ii), enlarging X, we may assume that for each $[p] - [q] \in \mathcal{P}$ there are $k, n \in \mathbb{N}$ such that p and q is in the C^*-subalgebra of $A \otimes M_k(\mathbb{C}) \otimes O_{n+1}$ generated by elementary tensors of the form $a \otimes m \otimes o$ with $a \in X$, $m \in M_k(\mathbb{C})$ and $o \in O_{n+1}$ (or with the O_{n+1} omitted as appropriate). Part (ii) follows on noting that for any such $[p] - [q]$ we can force

$$\|(\psi \otimes \mathrm{id})(p) - \phi \otimes \mathrm{id})(p)\| < 1/10$$

by choosing ϵ and γ small enough (here we use Remark 2.3), and similarly for q. ☐

3.3 The approximate K-homology UCT

The Kasparov product defines a pairing

$$K^0(A) \to \mathrm{Hom}_\Lambda(\underline{K}_0(A), \underline{K}_0(\mathbb{C}))$$

of the usual K-homology group of a C^*-algebra A with its total K-theory as in Examples 3.14. This is continuous when $\mathrm{Hom}_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(\mathbb{C}))$ is equipped with the topology of pointwise convergence: indeed, this a special case of continuity of the Kasparov product [40, Theorem 3.5]. As the topology of pointwise convergence on $\mathrm{Hom}_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(\mathbb{C}))$ is Hausdorff, this pairing necessarily sends the closure $\overline{\{0\}}$ of the trivial subgroup to zero, and so descends to a pairing

$$\Psi : KL^0(A) \to \mathrm{Hom}_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(\mathbb{C})) \tag{17}$$

of the KL-homology group of line (7) with total K-theory.

We now need to appeal to the universal coefficient theorem of Rosenberg and Schochet [123]. We will not need the UCT at its full strength, 'just' the property in the next definition.

Definition 3.22. A separable unital C^*-algebra A satisfies the *approximate K-homology UCT* if the continuous homomorphism Ψ from line (17) is a homeomorphism.

The following theorem is a special case of a result of Dadarlat [40, Theorem 4.1].

Theorem 3.23 (Dadarlat)**.** *Let A be a separable, unital C^*-algebra that satisfies the UCT. Then A satisfies the approximate K-homology UCT of Definition 3.22.* □

Remark 3.24. The reader might compare the approximate K-homology UCT to the *Approximate UCT* (AUCT) introduced by Lin in [97] as a bivariant version of Definition 3.22 above. Dadarlat showed the AUCT to be equivalent to the full UCT for a nuclear C^*-algebra A in [40, Theorem 5.4]. Definition 3.22 can be thought of as the 'AUCT for K-homology' (as opposed to for full bivariant KK-theory).

Essentially the same argument as for Theorem 3.23 shows that the approximate K-homology UCT is implied by Brown's K-homology UCT [26], i.e. there are general implications

$$\text{UCT} \;\Rightarrow\; K\text{-homology UCT} \;\Rightarrow\; \text{approximate } K\text{-homology UCT}.$$

We do not know if either implication is reversible (although it seems unlikely). There are examples of C^*-algebras that do not satisfy the UCT: the first such examples come from [128, Sections 4 and 5], and these are still probably the most tractable C^*-algebras known to fail the UCT. I do not know if there are C^*-algebras that do not satisfy the K-homology UCT.

Having got through these preliminaries, our last aim in this subsection is to record the following result. The proof is essentially bookkeeping, and more-or-less follows from results already in the literature; unfortunately, however, the isomorphism Θ of Theorem 3.8 is not defined very directly, so there is quite a lot of bookkeeping to do.

Theorem 3.25. *With Θ, Φ and Ψ as in Theorem 3.8, Proposition 3.19, and line (17) respectively, the diagram below commutes*

$$\begin{array}{ccc} KL^0(A) & \xrightarrow{\;\;\Theta\;\;} & \lim\limits_{\leftarrow} K^0_\epsilon(X) \\ & \searrow^{\Psi} & \downarrow \Phi \\ & & Hom_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(\mathbb{C})) \end{array} .$$

Moreover, all the maps are continuous, and Θ is a homeomorphism. If also A satisfies the approximate K-homology UCT of Definition 3.22 then Φ and Ψ are also homeomorphisms.

We need one more lemma before we get to the proof of this.

Lemma 3.26. *Let B be a stable*[21] *C^*-algebra. Let*

$$\mathfrak{A}(B) := \frac{C_b([1,\infty), B)}{C_0([1,\infty), B)}$$

be the asymptotic algebra in the sense of Connes-Higson [36]. Then the inclusion of B in $\mathfrak{A}(B)$ as constant functions induces an isomorphism on K-theory.

Moreover, an inverse to this isomorphism can be defined as follows. First, one shows that any element of $K_0(\mathfrak{A}(B))$ can be represented in the form $[p] - [\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]$, where p is a projection in the 2×2 matrices over the unitization of $\mathfrak{A}(B)$ whose difference with $(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$ is in $M_2(\mathfrak{A}(B))$. Let a be a self-adjoint element of the 2×2 matrices over the unitization of $C_b([1,\infty), B)$ that lifts p and is such that $a - (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix})$ is in $M_2(C_0([1,\infty), B)$. Let χ be the characteristic function of $(1/2, \infty)$. Then the inverse we want can be given by

$$[p] - [\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}] \mapsto [\chi(a(t))] - [\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}]$$

for any sufficiently large $t \in [1, \infty)$.

If we instead define

$$\mathfrak{A}_u(B) := \frac{C_{ub}([1,\infty), B)}{C_0([1,\infty), B)}$$

where $C_{ub}([1,\infty), B)$ denotes uniformly continuous and bounded functions from $[1,\infty)$ to B, then the same statements hold with $\mathfrak{A}_u(B)$ in place of $\mathfrak{A}(B)$.

[21] We warn the reader that something like the stability assumption is necessary. For example, the conclusion fails if $B = \mathbb{C}$.

Proof. The C^*-algebra $C_0([1,\infty),B)$ is contractible, so for the isomorphism result it suffices to show that the constant inclusion $B \to C_b([1,\infty),B)$ induces an isomorphism on K-theory. For this, it suffices to show that the evaluation-at-one map $C_b([1,\infty),B) \to B$ induces an isomorphism on K-theory, and for this it suffices to show that $K_*(I) = 0$, where $I := \{f \in C_b([1,\infty),B) \mid f(1) = 0\}$ is the kernel of that map.

Let then

$$I_O := C_0((1,2],B)\prod_{n=1}^{\infty} C([2n{+}1,2n{+}2],B), \quad I_E := \prod_{n=1}^{\infty} C([2n,2n{+}1],B)$$

and

$$I_Z := \prod_{n=2}^{\infty} B.$$

Then there is a pullback square of C^*-algebras and canonical restriction morphisms

$$\begin{array}{ccc} I & \longrightarrow & I_O \\ \downarrow & & \downarrow \\ I_E & \longrightarrow & I_Z \end{array} \tag{18}$$

giving rise to a Mayer-Vietoris sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(I_O)\oplus K_0(I_E) & \longrightarrow & K_0(I_Z) \\ \uparrow & & & & \downarrow \\ K_1(I_Z) & \longleftarrow & K_1(I_O)\oplus K_1(I_E) & \longleftarrow & K_1(I) \end{array} \tag{19}$$

where the horizontal maps are induced by the morphisms in line (18) (see for example [140, Definition 2.7.14 and Proposition 2.7.15]). On the other hand, using stability of B, the K-theory of the direct products splits

$$K_i(I_O) \cong K_i(C_0((1,2],B))\prod_{n=1}^{\infty} K_i(C([2n+1,2n+2],B))$$

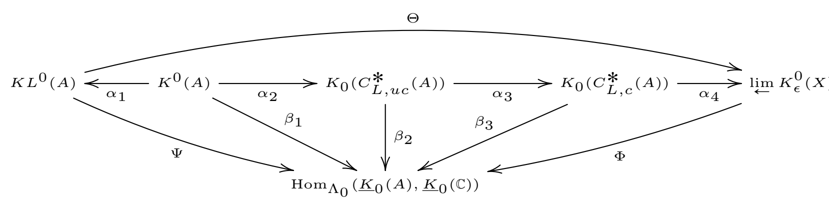
and similarly for I_E and I_Z (see for example [140, Proposition 2.7.12]). The required vanishing result follows readily from this and a computation based on the Mayer-Vietoris sequence in line (19).

For the construction of the inverse, it suffices to show that the given process induces a well-defined map on K-theory that is a one-sided inverse to the constant inclusion. This is straightforward: we leave the details to the reader.

Finally, the uniformly continuous case follows from exactly the same argument (or an argument based on an Eilenberg swindle: compare [140, page 229]). ☐

Proof of Theorem 3.25. Continuity of Ψ is a special case of [40, Theorem 3.5] as already noted above. Continuity of Φ is contained in Proposition 3.19. Moreover, that Θ is a homeomorphism is covered in Theorem 3.8 above. On the other hand, if A satisfies the approximate K-homology UCT of Definition 3.22, then Ψ is a homeomorphism by definition. Hence Φ will also be a homeomorphism in that case once we have established commutativity; this is the rest of the proof.

For commutativity, we use the following diagram



(20)

that we explain now.

We start with the top part of the diagram. The map labeled α_1 is the canonical quotient map from K-homology to KL-homology. The C^*-algebra $C^*_{L,c}(A)$ (respectively, $C^*_{L,uc}(A)$) consists of bounded and continuous (respectively, uniformly continuous) functions $b : [1, \infty) \to \mathcal{K}(H)$ such that $[b(t), a] \to 0$ as $t \to \infty$ for all $a \in A$. The map labeled α_2 is defined and shown to be an isomorphism in [49, Theorem 4.5][22]. The map labeled α_3 is the canonical inclusion, and is shown to be an isomorphism in [139, Theorem 3.4]. The map labeled α_4 is defined by first showing that every element of $C^*_{L,c}(A)$ can be represented in the form $[p] - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for some projection p in $M_2(C^*_{L,c}(A)^+)$ (here $C^*_{L,c}(A)^+$ is the unitization of $C^*_{L,c}(A)$) such that $p - \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \in M_2(C_{L,c}(A))$; then choosing $t_{X,\epsilon} \in [1, \infty)$ for each (X, ϵ) such that $\|[p_t, x]\| < \epsilon$ for all $t \geqslant t_{X,\epsilon}$ and all $x \in X$ and sending $[p] - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ to the element $([p_{t_{X,\epsilon}}])_{(X,\epsilon)\in\mathcal{I}_A}$ of the inverse limit.

Now, the proofs of [139, Theorem 4.14 and Proposition A.18] give that the map $\alpha_4 \circ \alpha_3 \circ \alpha_2 : K^0(A) \to \varprojlim K^0_\epsilon(X)$ descends to an isomorphism on K-homology; this is exactly the map we are calling Θ, and therefore the top part of the diagram commutes.

Let us now look at the lower triangles in diagram (20). The map labeled β_1 is the standard pairing between K-homology and total K-theory; as the map Ψ is by definition the map this induces on KL-homology, the left-most triangle in diagram (20) commutes.

The maps labeled β_2 and β_3 are versions with coefficients of the map defined by Wang, Zhang, and Zhou in [135, Definition 3.6] that we now

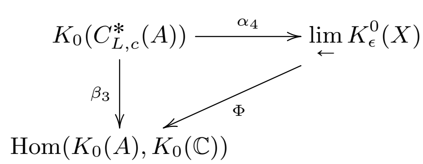
[22] This is slightly off: the paper [49] uses non-unital absorbing representations; it is not difficult to show that for unital C^*-algebras one may instead use a unital absorbing representation, and that this makes no difference on the level of K-theory, however. This slight elision occurs at a few other points in the proof below, and is not a serious issue anywhere it appears.

spell out. For notational simplicity, however, we just do the pairing with $K_0(A)$: the same argument works with "$\cdot \otimes O_{n+1}$" dragged through the proof for the groups with finite coefficients. It is straightforward to see that [135, Definition 3.6] is equivalent to the following description of the map β_2. Any class in $K_0(C^*_{L,c}(A))$ can be represented in the form $[p] - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ where $p = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + b$ for some $b \in M_2(C^*_{L,c}(A))$. Let $q \in M_k(A)$ be a projection representing a class in $K_0(A)$. Then the element $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + 1_{M2} \otimes qb \otimes 1_{M_k}$ defines a projection in 2×2 matrices over the unitization of the asymptotic algebra $\mathfrak{A}_u(M_k(\mathcal{K}(H)))$ of Lemma 3.26 whose difference with $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right)$ is in $\mathfrak{A}_u(M_k(\mathcal{K}(H)))$. The map β_2 then takes $[p] - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ to the mage of the class

$$\left[\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) + 1_{M2} \otimes qb \otimes 1_{M_k}\right] - \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] \in K_0(\mathfrak{A}(\mathcal{K}(H))$$

under the isomorphism $K_0(\mathfrak{A}_u(\mathcal{K}(H)) \cong K_0(\mathcal{K}(H)) \cong K_0(\mathbb{C})$ from Lemma 3.26. The map β_3 is defined entirely analogously, but with $\mathfrak{A}_u(\mathcal{K}(H))$ replaced by $\mathfrak{A}(\mathcal{K}(H))$.

Now the triangle in diagram (20) involving α_3, β_2 and β_3 commutes. On the other hand, the triangle involving β_1, α_2 and β_2 commutes by [135, Theorem 3.7]. It remains at this point to show that the diagram



commutes (and similarly for the case with finite coefficients). This follows from the description of the isomorphism of $K_0(\mathfrak{A}(\mathcal{K}(H)) \to K_0(\mathcal{K}(H))$ given in Lemma 3.26, the description of β_3 above, and the definition of Φ from Lemma 3.19, so we are done. □

Remark 3.27. As we will need it later, let us spell out in a little more detail what it means for the map

$$\Phi : \varprojlim K^0_\epsilon(X) \to \mathrm{Hom}_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(\mathbb{C}))$$

to be a homeomorphism; this is quite a powerful statement that will be very useful later. A neighborhood basis of the zero element in $\varprojlim K^0_\epsilon(X)$ is given by the subgroups $U_{Y,\delta}$ consisting of all elements in $\varprojlim K^0_\epsilon(X)$ that map to zero in $K^0_\delta(Y)$ as (Y, δ) ranges over the set $\mathcal{I}_A$ of Definition 3.4. A neighborhood basis of the zero map in $\mathrm{Hom}_{\Lambda_0}(\underline{K}_0(A), \underline{K}_0(\mathbb{C}))$ consists of the subsets V_P of elements that restrict to the zero map P as P ranges over all K-data as in Definition 3.20.

Now, Φ being continuous means that for any $(Y, \delta) \in \mathcal{I}_A$ there exists a K-datum P as above such that Φ descends to a well-defined map

$$K^0_\delta(Y) \to \mathrm{Hom}_{\Lambda_0}(P, \underline{K}_0(\mathbb{C})).$$

Indeed, the map above is the composition

$$K^0_\delta(Y) \overset{\Phi^{(Y,\delta)}}{\to} \mathrm{Hom}_{\Lambda_0}(\underline{K}^\delta_0(Y), \underline{K}_0(\mathbb{C})) \overset{\kappa_P^{(Y,\delta)}}{\to} \mathrm{Hom}_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$$

of the maps of Proposition 3.19 and Lemma 3.21.

On the other hand, and much less concretely, the fact that Φ^{-1} is continuous means that for any K-datum P there exists $(Y,\delta) \in \mathcal{I}_A$ such that Φ^{-1} descends to a well-defined map

$$(\Phi^{-1})^P_{(Y,\delta)} : \mathrm{Hom}_{\Lambda_0}(P, \underline{K}_0(\mathbb{C})) \to K^0_\delta(Y).$$

Finally these maps are consistent with the inverse limit structures when they are defined, i.e. we have commutative diagrams of set maps

$$\begin{array}{ccccc} & & K^0_\delta(Y) & & \\ & \overset{(\Phi^{-1})^Q_{(Y,\delta)}}{\nearrow} & & \overset{\kappa_P^{(Y,\delta)}\circ\Phi^{(Y,\delta)}}{\searrow} & \\ \mathrm{Hom}_{\Lambda_0}(Q, \underline{K}_0(\mathbb{C})) & & \longrightarrow & & \mathrm{Hom}_{\Lambda_0}(P, \underline{K}_0(\mathbb{C})) \end{array}$$

and

$$\begin{array}{ccccc} K^0_\delta(Y) & & \longrightarrow & & K^0_\epsilon(X) \\ & \underset{\kappa_P^{(Y,\delta)}\circ\Phi^{(Y,\delta)}}{\searrow} & & \underset{(\Phi^{-1})^P_{(X,\epsilon)}}{\nearrow} & \\ & & \mathrm{Hom}_{\Lambda_0}(P, \underline{K}_0(\mathbb{C})) & & \end{array}$$

where defined (here the horizontal maps are forgetful maps).

In summary, the fact that Φ is a homeomorphism on the inverse limits means that it comes from a coherent system of maps between the inverse systems. For Φ this is essentially true by definition; the content is to get it also for Φ^{-1}.

4 Stable uniqueness for representations of C^*-algebras

Our goal in this section is to establish a stable[23] uniqueness theorem, following Dadarlat-Eilers [46, Section 4] and Lin [96]. Very roughly, such a theorem says that given two ucp approximately multiplicative maps between C^*-algebras, if they agree on a large enough subset of K-theory, then they are approximately unitarily equivalent after taking block sum with another homomorphism.

We have not attempted to state our stable uniqueness result in maximal generality. For example, a version for the bivariant theory $KL(A,B)$

[23] "Stable" is used in the sense of K-theory: i.e. 'up to taking block sum with something'. It is not directly connected to representation stability as in Definition 1.4.

should be possible under appropriate assumptions (for our methods to work, these probably have to include nuclearity of B), but we will not pursue that here.

The following definition is a slight variant on [46, Definition 3.7].

Definition 4.1. Let A be a unital C^*-algebra. A $\underline{K}_0$*-triple* for A is a triple (P, X, ϵ) where:

(i) X is a finite self-adjoint subset of the unit ball of A;

(ii) ϵ is a positive real number;

(iii) P is a K-datum as in Definition 3.20 such that the map

$$\kappa_P^{(X,\epsilon)} : \mathrm{Hom}_{\Lambda_0}(\underline{K}_0^{\epsilon}(X), \underline{K}_0(\mathbb{C})) \to \mathrm{Hom}_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$$

of Lemma 3.21 above is well-defined.

Here is the main theorem of this section.

Theorem 4.2. *Let A be a separable, unital C^*-algebra. Assume that A is RFD as in Definition 1.12, and satisfies the approximate K-homology UCT of Definition 3.22.*

Let $\mathcal{X}$ be a family of finite, symmetric subsets of the unit ball of A that is closed under finite unions, and such that $\bigcup_{X\in\mathcal{X}} X$ generates A as a C^*-algebra Then for any $X \in \mathcal{X}$ and $\epsilon > 0$ there exists a $\underline{K}_0$-triple (P, Y, δ) with $Y \in \mathcal{X}$ and with the following property.

Let $\phi : A \to M_n(\mathbb{C})$ be a ucp (Y, δ)-$*$-homomorphism as in Definition 3.2, and let

$$\kappa_P^{(Y,\delta)}(\phi_*) \in Hom_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$$

be the element given by Lemma 3.21 and the definition of a $\underline{K}_0$-triple. Let $\pi : A \to M_n(\mathbb{C})$ be a $*$-representation, and assume that $\pi_* = \kappa_P^{(Y,\delta)}(\phi_*)$ as elements of $Hom_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$.

Then there exist a representation $\theta : A \to M_k(\mathbb{C})$ and a unitary $u \in M_{n+k}(\mathbb{C})$ such that

$$\|u(\phi(x) \oplus \theta(x))u^* - (\pi(x) \oplus \theta(x))\| < \epsilon$$

for all $x \in X$.

Remark 4.3. In general, one can always take $\mathcal{X}$ as in the theorem to be the set of all finite symmetric subsets of the unit ball (or unit sphere) of A, and we could have just stated the theorem in this way. The purpose of including the collection $\mathcal{X}$ in the statement is to strengthen the conclusion in special cases where a good choice of $\mathcal{X}$ is possible. For example, if A is finitely generated by some set symmetric subset X of its unit ball, one can take $\mathcal{X} = \{X\}$. Importantly for us, if $A = C^*(\Gamma)$ for a group Γ, we can take $\mathcal{X}$ to consist of all finite symmetric subsets of Γ, or to be a singleton consisting of a finite symmetric generating set of Γ (if one exists).

As already mentioned, Theorem 4.2 is inspired by work of Lin [96, Theorem 4.8] and Dadarlat-Eilers [46, Theorem 4.15]. Theorem 4.2 above has the advantage over these earlier results that it requires no nuclearity or exactness assumptions on A. This is important for our applications as group C^*-algebras of non-amenable RFD groups are never exact: see [85, Proposition 7.1] or [28, Proposition 3.7.11]. On the other hand, Theorem 4.2 has the disadvantage over the work of Lin and Dadarlat-Eilers that there is no control over the size of the 'auxiliary representation' θ.

We need some technical lemmas before we get to the proof of Theorem 4.2. The first is well-known; we could not find exactly what we need in the literature so give a proof. Some of the argument is quite similar to that from Proposition 2.2 part (iii)

Lemma 4.4. *Let A be a separable, unital C^*-algebra, let X be a finite, symmetric subset of the unit ball of A, and let $\epsilon > 0$. Let H be a separable Hilbert space, and let A be included in $\mathcal{B}(H)$ via a faithful unital representation such that $A \cap \mathcal{K}(H) = \{0\}$. Let $\phi : A \to M_n(\mathbb{C})$ be a ucp (X, δ)-$*$-homomorphism in the sense of Definition 3.2.*

Then there are a rank n projection $p \in \mathcal{B}(H)$, and a unitary isomorphism $u : \mathbb{C}^n \to range(p)$ such that

$$\|u^* papu - \phi(a)\| < \epsilon \quad \textit{and} \quad \|[p, a]\| < 2\sqrt{3\epsilon + \delta}.$$

for all $a \in X$.

Proof. The finite-dimensional version of Voiculescu's theorem (see for example [28, Proposition 1.7.1]) gives a rank n projection $p \in \mathcal{B}(H)$ and a unitary $u : \mathbb{C}^n \to \text{range}(p)$ such that $\|u^* papu - \phi(a)\| < \epsilon$ holds for all $a \in X \cup X^2$. We claim such a p automatically satisfies $\|[p, a]\| < 4\sqrt{\delta}$ for all $a \in X$, which will complete the proof.

Write $\psi(a) = u\phi(a)u^*$, and note that ψ is an (X, ϵ)-$*$-homomorphism. Then for any $a \in X$,

$$\begin{aligned}
\|pa(1-p)a^*p\| &= \|paa^*p - papa^*p\| \\
&\leqslant \|paa^*p - u\phi(aa^*)u^*\| + \|papa^*p - u\phi(a)u^*u\phi(a^*)u^*\| \\
&\leqslant \|\psi(aa^*) - \psi(a)\psi(a^*)\| + 3\epsilon \\
&< 3\epsilon + \delta
\end{aligned}$$

(we use that $X = X^*$, so both a and a^* are in A, and aa^* is in X^2). Hence by the C^*-identity, $\|pa(1-p)\| < \sqrt{3\epsilon + \delta}$ for all $a \in X$, and similarly, $\|(1-p)ap\| < \sqrt{3\epsilon + \delta}$ for all $a \in X$. Hence for any $a \in X$,

$$\|pa - ap\| = \|pa(1-p) - (1-p)ap\| < 2\sqrt{3\epsilon + \delta}.$$

and we are done. ∎

The next result is again more-or-less folklore: the precise statement we give can be found in [141, Lemma 2.20].

Lemma 4.5. *Let $\delta \in (0, 1/2)$, let a be a self-adjoint element in a C^*-algebra A whose spectrum does not intersect $(\delta, 1-\delta)$, and let χ be the characteristic function of $(1/2, \infty)$. Then for any $b \in A$,*

$$\|[\chi(a), b]\| \leqslant \frac{1}{1-2\delta}\|[a, b]\|. \qquad \square$$

The next lemma is a version of the basic fact from operator K-theory that nearby projections are unitary equivalent (see for example [122, Proposition 2.2.6]), that also gives control on commutators. The proof is the same as the standard one with a little extra work to keep track of commutator estimates throughout the construction.

Lemma 4.6. *Let A be a unital C^*-algebra, let X be a finite, symmetric subset of the unit ball of A, and let $\epsilon > 0$. Let p, q be projections in A such that $\|p-q\| < 1/4$, and such that $\|[p, x]\| < \epsilon$ and $\|[q, x]\| < \epsilon$ for all $x \in X$. Then there is a unitary $u \in A$ such that $upu^* = q$ and $\|[u, x]\| < 28\epsilon$ for all $x \in X$.*

Proof. Define $v := pq + (1-p)(1-q)$ and note that $vq = pv$ and that $\|[v, x]\| < 4\epsilon$ for all $x \in X$. Moreover

$$\|1 - v\| = \|(2q-1)(p-q)\| = \|p-q\| < 1/4,$$

whence v is invertible, $\|v\| < 5/4$ and

$$\|1 - v^*v\| \leqslant \|(1-v^*)v\| + \|1-v\| < 9/16$$

whence

$$v^*v \geqslant 7/16. \tag{21}$$

Define now $u := v(v^*v)^{-1/2}$, which is unitary. As p and q are self-adjoint, we see that $qv^* = v^*p$ and so v^*v commutes with p and q. Hence $(v^*v)^{-1/2}$ also commutes with p and q, and so $upu^* = q$. It remains to estimate $\|[u, x]\|$ for each $x \in X$. For this, we have

$$\begin{aligned} \|[u, x]\| &\leqslant \|[v, x]\|\|(v^*v)^{-1/2}\| + \|v\|\|[x, (v^*v)^{-1/2}]\| \\ &< 4\epsilon \cdot 2 + (5/4) \cdot \|[x, (v^*v)^{-1/2}]\| \end{aligned} \tag{22}$$

by the estimates already noted. To estimate $\|[x, (v^*v)^{-1/2}]\|$ we use the identity

$$t^{-1/2} = \frac{2}{\pi}\int_0^\infty (\lambda^2 + t)^{-1} d\lambda,$$

valid for any $t > 0$. Using the identity $[x, a^{-1}] = a^{-1}[a, x]a^{-1}$ we see that

$$\|[x, (v^*v)^{-1/2}]\| \leqslant \frac{2}{\pi}\int_0^\infty \|(\lambda^2 + v^*v)^{-1}\|\|[v^*v, x]\|\|(\lambda^2 + v^*v)^{-1}\| d\lambda.$$

Using line (21), we see that $\|(\lambda^2 + v^*v)^{-1}\| \leqslant (\lambda^2 + 7/16)^{-1}$, and so the above line implies that

$$\|[x, (v^*v)^{-1/2}]\| \leqslant \frac{2\|[x, v^*v]\|}{\pi} \int_0^\infty (\lambda^2 + 7/16)^{-2} d\lambda.$$

Computing, the integral in the line above equals $(2\pi)/\sqrt{7}$, and $\|[x, v^*v]\| < 10\epsilon$, so we get

$$\|[x, (v^*v)^{-1/2}]\| < 16\epsilon.$$

Substituting this back into line (22) gives $\|[u, x]\| < 28\epsilon$ as claimed. ☐

The next lemma is essentially contained in [141, Section 4.4]: see in particular [141, Proposition 4.17]. As that reference does not contain exactly what we need, we sketch a proof.

Lemma 4.7. *Let A be a unital C^*-algebra, let X be a finite, symmetric subset of the unit ball of A, and let $\epsilon > 0$. Let $(p_t)_{t\in[0,1]}$ be a path of projections in A such that $\|[p_t, x]\| < \epsilon$ for all $t \in [0, 1]$. Then there exists n and a unitary $u \in M_{2n+1}(A) = A \otimes M_{2n+1}(\mathbb{C})$ such that*

$$u(p_0 \oplus 1_n \oplus 0_n)u^* = p_1 \oplus 1_n \oplus 0_n \tag{23}$$

and $\|[u, x \otimes 1_{2n+1}]\| < 4 \cdot 10^4\epsilon$ for all $x \in X$.

Proof. Following [141, Lemma 4.15], there is n and a 16-Lipschitz path $(q_t)_{t\in[0,1]}$ of projections between $p_0 \oplus 1_n \oplus 0_n$ and $p_1 \oplus 1_n \oplus 0_n$ in $M_{2n+1}(A)$ with the property that $\|[x, q_t]\| < 21\epsilon$ for all $x \in X$ (here and throughout we abuse notation and write "x" in place of "$x \otimes 1_{2n+1}$"). Hence we may find $0 = t_0 < t_1 < \cdots < t_{65} = 1$ such that $\|q_{t_i} - q_{t_{i+1}}\| < 1/4$ for all $i \in \{0, ..., 64\}$. Lemma 4.6 then gives unitaries u_i such that $u_i q_{t_i} u_i^* = q_{t_{i+1}}$ and so that

$$\|[u_i, x]\| < 588\epsilon \tag{24}$$

for all $x \in X$ and all $i \in \{0, ..., 64\}$. Define $u = u_{64}u_{63} \cdots u_1u_0$. Then u is a unitary that satisfies the equation in line (23), and we have that for any $x \in X$,

$$\|[u, x]\| \leqslant \sum_{i=0}^{64} \|[u_i, x]\|$$

by repeated use of the Leibniz rule. Combined with line (24), this implies the desired estimate. ☐

Proof of Theorem 4.2. Let X and ϵ be as in the statement; we may assume $\epsilon < 1$. The discussion in Remark 3.27 implies that there exists a $\underline{K}_0$-triple (P, Y, δ) such that if $[p], [q] \in K^0_{5\sqrt{\delta}}(Y)$ are such that (with notation as in Proposition 3.19 and Lemma 3.21)

$$\kappa_P^{(Y,5\sqrt{\delta})}(\Phi^{(Y,5\sqrt{\delta})}(p)) = \kappa_P^{(Y,5\sqrt{\delta})}(\Phi^{(Y,5\sqrt{\delta})}(q))$$

then the images of $[p]$ and $[q]$ under the forgetful map

$$K^0_{5\sqrt{\delta}}(Y) \to K^0_{\epsilon/(8\cdot 10^5)}(X)$$

of Remark 3.6, part (ii) are the same. Using that the collection $\{(Z,\gamma) \mid Z \in \mathcal{X}, 0 < \gamma < \epsilon/4\}$ is cofinal in the directed set $\mathcal{I}_A$ of Definition 3.4, we may moreover assume that Y is in $\mathcal{X}$ and $\delta < \epsilon/4$. We claim this Y and δ work.

Let $\phi : A \to M_n(\mathbb{C})$ be a ucp (Y,δ)-$*$-homomorphism. Let $\pi : A \to M_n(\mathbb{C})$ be a $*$-representation as in the statement, so in particular

$$\kappa_P^{(Y,\delta)}(\phi_*) = \pi_*. \tag{25}$$

As A is separable and RFD there is a countable family of finite-dimensional unital representations of A that separates points and contains π. Take the direct sum of this family, then take the direct sum of that representation with itself countably many times. We use this representation to identify A with a unital C^*-subalgebra of $\mathcal{B}(H)$ (with H the separable Hilbert space underlying the given representation), and to define the controlled K-homology groups of Definition 3.1.

Let $\gamma > 0$ be as in the conclusion of Lemma 3.21 part (ii) for $(Y, 5\sqrt{\delta})$; we may assume that $\gamma < \delta$. Then Lemma 4.4 gives a rank n projection $p_\phi \in \mathcal{B}(H)$ such that $\|[p_\phi, a]\| < 4\sqrt{\delta}$ for all $a \in Y$, and a unitary isomorphism $w : \mathbb{C}^n \to \text{range}(p_\phi)$ such that $\|w^* p_\phi a p_\phi w - \phi(a)\| < \gamma/2$ for all $a \in X$. As the representation $A \subseteq \mathcal{B}(H)$ is a direct sum of finite-dimensional representations, we may assume on perturbing p_ϕ that it is dominated by a finite rank projection r_ϕ in the commutant of A, and that the perturbed projection satisfies

$$\|[p_\phi, a]\| < 5\sqrt{\delta} \tag{26}$$

and

$$\|w^* p_\phi a p_\phi w - \phi(a)\| < \gamma \tag{27}$$

for all $a \in Y$. Define

$$\phi' : A \to \mathcal{B}(p_\phi H), \quad \phi'(a) := p_\phi a p_\phi \tag{28}$$

to be the ucp map given by compression by p_ϕ. Using line (27) and the choice of γ, we have that

$$\kappa_P^{(Y,\delta)}(\phi_*) = \kappa_P^{(Y,5\sqrt{\delta})}(\phi_*) = \kappa_P^{(Y,5\sqrt{\delta})}(\phi'_*) \tag{29}$$

(here we use that $5\sqrt{\delta} \geqslant \delta$ to think of ϕ as both a (Y,δ)-$*$-homomorphism and as a $(Y, 5\sqrt{\delta})$-$*$-homomorphism; the first equality above then holds by definition).

Define

$$p := \begin{pmatrix} 1 & 0 \\ 0 & p_\phi \end{pmatrix} \in M_2(\mathcal{B}(H))$$

and note that by line (26), p defines an element of the set $\mathcal{P}_{5\sqrt{\delta}}(Y)$ of Definition 3.1. On the other hand, recalling that π is a subrepresentation of the representation we are using to identify A with a C^*-subalgebra of $\mathcal{B}(H)$, there is a rank n projection q_π in the commutant of A such that compression by q_π identifies with π. Then

$$q := \begin{pmatrix} 1 & 0 \\ 0 & q_\pi \end{pmatrix} \in M_2(\mathcal{B}(H))$$

is also an element of $\mathcal{P}_{5\sqrt{\delta}}(Y)$. Moreover the elements $\kappa_P^{(Y,5\sqrt{\delta})}(\Phi^{(Y,5\sqrt{\delta})}(p))$ and $\kappa_P^{(Y,5\sqrt{\delta})}(\Phi^{(Y,5\sqrt{\delta})}(q))$ agree with $\kappa_P^{(Y,5\sqrt{\delta})}(\phi'_*)$ as in line (28) and with $\kappa_P^{(Y,5\sqrt{\delta})}(\pi_*)$ respectively (by definition - see Proposition 3.19 and line (14) above). Hence by lines (25) and (29) we have that

$$\kappa_P^{(Y,5\sqrt{\delta})}(\Phi^{(Y,5\sqrt{\delta})}(p)) = \kappa_P^{(Y,5\sqrt{\delta})}(\Phi^{(Y,5\sqrt{\delta})}(q))$$

in

$$\mathrm{Hom}_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$$

and therefore by choice of (Y, δ), the images of $[p]$ and $[q]$ under the forgetful map

$$K^0_{5\sqrt{\delta}}(Y) \to K^0_{\epsilon/(8\cdot 10^5)}(X)$$

of Remark 3.6, part (ii) are the same.

Now, with notation as in Definition 3.1 there is a homotopy $(p_t)_{t\in[0,1]}$ passing through the space $\mathcal{P}_{\epsilon/8\cdot 10^5}(X)$ of Definition 3.1 and connecting p to q. As the image of this homotopy is compact, and as the original representation on H is a direct sum of finite-dimensional representations, and as p_ϕ and q_π are dominated by finite rank projections in the commutant of A, there is a finite rank projection $r \in \mathcal{B}(H)$ that commutes with A, that dominates q_π and p_ϕ, and is such that

$$\left\| \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} \left(p_t - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) - \left(p_t - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \right\| < \frac{\epsilon}{4 \cdot 10^5}$$

for all $t \in [0,1]$. Rearranging the above inequality, we see that

$$\left\| \begin{pmatrix} 1-r & 0 \\ 0 & 1-r \end{pmatrix} p_t - \begin{pmatrix} 1-r & 0 \\ 0 & 0 \end{pmatrix} \right\| < \frac{\epsilon}{4 \cdot 10^5}$$

Taking adjoints gives

$$\left\| p_t \begin{pmatrix} 1-r & 0 \\ 0 & 1-r \end{pmatrix} - \begin{pmatrix} 1-r & 0 \\ 0 & 0 \end{pmatrix} \right\| < \frac{\epsilon}{4 \cdot 10^5}$$

and combining the last two displayed lines gives

$$\left\| p_t \begin{pmatrix} 1-r & 0 \\ 0 & 1-r \end{pmatrix} - \begin{pmatrix} 1-r & 0 \\ 0 & 1-r \end{pmatrix} p_t \right\| < \frac{\epsilon}{2 \cdot 10^5}.$$

Hence $\|[p_t, (\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix})]\| < \epsilon/(2 \cdot 10^5)$, and so if $a_t := (\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix})p_t(\begin{smallmatrix} r & 0 \\ 0 & r \end{smallmatrix})$, then $\|a_t^2 - a_t\| < \epsilon/(2 \cdot 10^5) < 1/4$. In particular, if χ is the characteristic function of $(1/2, \infty)$, then χ is continuous on the spectrum of a_t. Moreover, as r commutes with A and as each p_t commutes with all elements of X up to $\epsilon/(2 \cdot 10^5)$, Lemma 4.5 gives that $\chi(a_t)$ commutes with all elements of X up to $\epsilon/10^5$. Write $\pi_r : A \to \mathcal{B}(H_r)$ for the (finite-dimensional) representation given by compression by r, we thus have a homotopy $(\chi(a_t))_{t \in [0,1]}$ between

$$\begin{pmatrix} 1_{H_r} & 0 \\ 0 & p_\phi \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1_{H_r} & 0 \\ 0 & q_\pi \end{pmatrix}$$

in $M_2(\mathcal{B}(H_r))$ that commutes with $(\pi_r \oplus \pi_r)(X)$ up to $\epsilon/10^5$.

Now, for each of notation, set C to be the unital C^*-algebra $M_2(\mathcal{B}(H_r))$, define $p_C := \left(\begin{smallmatrix} 1_{H_r} & 0 \\ 0 & p_\phi \end{smallmatrix}\right)$ and $q_C := \left(\begin{smallmatrix} 1_{H_r} & 0 \\ 0 & q_\pi \end{smallmatrix}\right)$, and write $\sigma : A \to C$ for the representation defined by compression by $r \oplus r$. Then Lemma 4.7 applied to the projections p_C and q_C gives $l \in \mathbb{N}$ and a unitary $M_{2l+1}(C))$ such that

$$v((p_\phi \oplus 1_{H_r}) \oplus 1_l \oplus 0_l)v^* = (q_\pi \oplus 1_{H_r}) \oplus 1_l \oplus 0_l) \tag{30}$$

and moreover so that

$$\|[v, \sigma(a)]\| < \epsilon/2 \tag{31}$$

for all $a \in X$ (we have also flipped the order of p_ϕ and 1_{H_r}, and similarly for q_π and 1_{H_r}).

Define now $p_0 := p_\phi \oplus 1_{H_r}) \oplus 1_l \oplus 0_l$ and $q_0 := (q_\pi \oplus 1_{H_r}) \oplus 1_l \oplus 0_l$. Set $u_0 = vp_0$, and let $\theta : A \to M_{2n+1}(C)$ be given by σ follows by compression by $0_{H_r} \oplus 1_{H_r} \oplus 1_l \oplus 0_l$. We regard moreover $\phi' : A \to C$ as given by σ followed by compression by $p_\phi \oplus 0_{H_r} \oplus 0_{2l}$, and $\pi : A \to C$ as given by σ followed by compression by $q_\pi \oplus 0_{H_r} \oplus 0_{2l}$. Then u is a unitary isomorphism between the Hilbert spaces underlying $\phi' \oplus \theta$ and $\pi \oplus \theta$, and we have moreover that for any $a \in X$,

$$\begin{aligned} \|u_0(\phi'(a) \oplus \theta(a))u_0^* - (\pi(a) \oplus \theta(a))\| &= \|vp_0\sigma(a)p_0v^* - q_0\sigma(a)q_0\| \\ &= \|q_0(v\sigma(a)v^* - \sigma(a))q_0\| \\ &\leqslant \|v\sigma(a)v^* - \sigma(a)\| \\ &< \epsilon/2 \end{aligned} \tag{32}$$

where the equality is from line (30), and the strict inequality is from line (31). Finally, let $u = u_0(w \oplus \mathrm{id}_\theta)$ where w is as in line (27) and id_θ is the identity on the Hilbert space underlying θ. Then

$$\begin{aligned} &\|u(\phi(a) \oplus \theta(a))u^* - (\pi(a) \oplus \theta(a))\| \\ &\quad \leqslant \|u_0((w\phi(a)w^* \oplus \theta(a)) - (\phi'(a) \oplus \theta(a))u_0^*\| \\ &\qquad + \|u_0(\phi'(a) \oplus \theta(a))u_0^* - (\pi(a) \oplus \theta(a))\| \\ &\quad \leqslant \|w\phi(a)w^* - \phi'(a)\| + \|u_0(\phi'(a) \oplus \theta(a))u_0^* - (\pi(a) \oplus \theta(a))\|. \end{aligned}$$

The estimates in line (27) (plus that $\gamma < \delta < \epsilon/2$) and line (32) complete the proof.

□

5 Stable uniqueness for representations of groups

Before getting to the main result on representation stability we one more technical lemmas.

Lemma 5.1. *Let Γ be a discrete group with the LLP as in Definition 2.4. Let S be a finite symmetric subset of Γ, let $\epsilon > 0$, and let P be a K-datum as in Definition 3.20 such that the homomorphism*

$$\kappa_P^{(S,\epsilon)} : Hom_{\Lambda_0}(\underline{K}_0^{\epsilon}(S), \underline{K}_0(\mathbb{C})) \to Hom_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$$

of Lemma 3.21 is defined. Then there is a finite subset T of Γ, $\delta > 0$, and $\gamma \in (0, \epsilon)$ with the following property.

Assume $\psi : \Gamma \to M_n(\mathbb{C})$ is a (T, δ)-representation as in Definition 1.3. Then exists a ucp (S, ϵ)-representation $\phi : \Gamma \to M_n(\mathbb{C})$ such that

$$\|\phi(s) - \psi(s)\| < \gamma \quad \textit{for all} \quad s \in S, \tag{33}$$

and moreover so that for any two ucp (S, ϵ)-representations $\phi, \phi' : \Gamma \to M_n(\mathbb{C})$ satisfying the estimate in line (33), the elements

$$\phi_*, \phi'_* \in Hom_{\Lambda_0}(\underline{K}_0^{\epsilon}(S), \underline{K}_0(\mathbb{C}))$$

defined using Proposition 2.2 part (ii) and Lemma 3.16 satisfy that

$$\kappa_P^{(S,\epsilon)}(\phi_*) = \kappa_P^{(S,\epsilon)}(\phi'_*)$$

as elements of $Hom_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$.

Proof. Let γ be as in Lemma 3.21 part (ii). Propositon 2.6 gives a finite subset T of Γ and $\delta > 0$ such that if $\psi : \Gamma \to \mathcal{B}(H)_1$ is a (T, δ)-representation, then there exists a ucp (S, ϵ)-representation $\phi : C^*(\Gamma) \to M_n(\mathbb{C})$ such that $\|\psi(s) - \phi(s)\| < \gamma$ for all $s \in S$. This gives the properties in the statement. □

Definition 5.2. With notation as in Lemma 5.1, for a fixed (S, ϵ) let (T, δ) be as in the conclusion, and let $\psi : \Gamma \to M_n(\mathbb{C})$ be a (T, δ)-representation. Then we write $\psi_* \in \mathrm{Hom}_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$ for the map $\kappa_P^{(X,\epsilon)}(\phi_*)$ for any ϕ_* as in the conclusion of Lemma 5.1.

Here is our main theorem. It is essentially just Theorem 4.2 specialized to the case where A is a group C^*-algebra, and $\mathcal{X}$ consists of unitary group elements, although there is a little more to say in the LLP case.

Theorem 5.3. *Let Γ be a countable discrete group such that $C^*(\Gamma)$ is RFD as in Definition 1.12 and satisfies the approximate K-homology UCT of Definition 3.22. Let $\mathcal{S}$ be a collection of finite subsets of Γ that is closed under finite unions, and with the property that $\bigcup_{S\in\mathcal{S}} S$ generates Γ*[24]*. Then for any $S \in \mathcal{S}$ and $\epsilon > 0$ there exist a $\underline{K}_0$-triple (P, T, δ) as in Definition 4.1 with $T \in \mathcal{S}$ and with the following property.*

Let $\phi : \Gamma \to M_n(\mathbb{C})$ be a ucp (T, δ)-representation in the sense of Definition 1.3, and let $\kappa_P^{(X,\epsilon)}(\phi_*) \in Hom_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$ be the element defined by ϕ using Lemma 3.21 and the definition of a $\underline{K}_0$-triple. Let $\pi : \Gamma \to M_n(\mathbb{C})$ be a unitary representation such that $\kappa_P^{(X,\epsilon)}(\phi_*) = \pi_*$ in $Hom_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$.

Then there is a unitary representation $\theta : \Gamma \to M_k(\mathbb{C})$ and a unitary $u \in M_{n+k}(\mathbb{C})$ such that

$$\|u(\phi(s) \oplus \theta(s))u^* - (\pi(s) \oplus \theta(s))\| < \epsilon$$

for all $s \in S$.

If moreover Γ has the LLP of Definition 2.4 and if $\mathcal{S}$ is the collection of all finite subsets of Γ, then one can drop the assumption that ϕ is ucp and use Definition 5.2 for the definition of ϕ_*.

Proof. Note that Proposition 2.2 part (ii) implies that a ucp representation $\phi : \Gamma \to M_n(\mathbb{C})$ extends uniquely by linearity and continuity to a ucp map $\phi : C^*(\Gamma) \to M_n(\mathbb{C})$, and that if $\phi : \Gamma \to M_n(\mathbb{C})$ is a ucp (T, δ)-representation, then the extended map $\phi : C^*(\Gamma) \to M_n(\mathbb{C})$ is a (T, δ)-$*$-homomorphism as in Definition 3.2 (note that ucp maps are automatically $*$-preserving). Having made this observation, the first two paragraphs of Theorem 5.3 are just Theorem 4.2 specialized to the case that $A = C^*(\Gamma)$ and the set $\mathcal{X}$ from Theorem 4.2 consists of unitaries coming from the group; .

The part with the additional assumption that $C^*(\Gamma)$ satisfies the LLP follows from the first part and Lemma 5.1. ☐

Remark 5.4. The condition in the theorem is in some sense also necessary for the conclusion to hold. To be more precise, the following holds.

"Let P be a K-datum as in Definition 3.20. Then there exists a finite subset S of Γ and $\epsilon > 0$ with the following property. Let T be a finite subset of Γ and $\delta > 0$ be such that the various components of P are contained in the image of the maps $K_0^\delta(T; \cdot) \to K_0(A; \cdot)$ of line (10) above; such exist by Lemma 3.17. Then if $\phi : \Gamma \to M_n(\mathbb{C})$ is a ucp (T, δ)-representation and $\pi : \Gamma \to M_n(\mathbb{C})$ is a unitary representation such that

[24] The most interesting case is when Γ is finitely generated and $\mathcal{S} = \{S\}$ for some finite generating set S; the reader should bear this case in mind.

ϕ_* and π_* do not agree on P, then for any representation $\theta : \Gamma \to M_k(\mathbb{C})$ and any unitary $u \in M_{n+k}(\mathbb{C})$,

$$\|u(\phi(s) \oplus \theta(s))u^* - (\pi(s) \oplus \theta(s))\| \geqslant \epsilon$$

for all $s \in S$."

This follows from basic continuity properties of K-theory and the fact that K-theory allows cancellations of block sums. This fundamental observation underlies many of the known obstructions to representation stability in operator norm: see for example [43] for far-reaching results in this direction.

In the next three remarks we say a little bit more about the acronym assumptions in Theorem 5.3: that the group C^*-algebra is RFD, UCT, and LLP.

Remark 5.5. We first look at the LLP assumption. The LLP was introduced for general C^*-algebras by Kirchberg [85, Section 2], and a wealth of information about the property and its connection to several deep conjectures can be found in the recent textbook [116].

Specializing to groups, we first note that all amenable groups have the LLP: indeed, for countable amenable groups this follows from the Choi-Effros lifting theorem [35] (see [5, Section 3] or [28, Theorem C.3] for simpler proofs) and nuclearity of the group C^*-algebra [92, Theorem 4.2] (see for example [28, Theorem 2.6.8] for a textbook treatment). This implies the LLP even for uncountable amenable groups, as any finite-dimensional subspace of $C^*(\Gamma)$ is contained in the C^*-subalgebra $C^*(\Gamma_0)$ for some countable subgroup. The class of groups with the LLP is closed under free products with finite amalgam [112, Comments below Proposition 3.21], and direct products with amenable groups: this follows from [85, Corollary 2.6 (iv)] and nuclearity of group C^*-algebras of amenable groups again. More generally, the class of groups with the LLP is closed under semidirect products with amenable groups: this follows from [30, Theorem 7.2 and Proposition 5.10] (see also [60, Corollary 8.5]).

The first examples of groups without the LLP were constructed by Thom [130, page 198], and more examples were exhibited by Ioana, Spaas, and Wiersma [76]; the latter examples include natural and well-studied groups like $\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$. It is an interesting open question whether the fundamental groups of surfaces of genus two, or of hyperbolic 3-manifolds, satisfy the LLP.

Remark 5.6. We now say some more about the UCT, and the approximate K-homology UCT assumption in Theorem 5.3. The UCT for K-homology (which is all we really need in this paper) was introduced by Brown in [26]. The stronger UCT for KK-theory was introduced later by Rosenberg and Schochet [123]; following standard practice in the literature, when we say "UCT" we mean the Rosenberg-Schochet KK-theory version.

The most prominent class of groups known to satisfy the UCT are the (countable) a-T-menable groups: this is a consequence of the work of Higson and Kasparov [73] on the Baum-Connes conjecture for such groups, as recorded by Tu [133, Proposition 10.7] (in a more general context). Another important class of UCT groups comes from Tu's *condition (BC′)* [132, Definition 2.5]: this is recorded in [105, Theorem 3.5][25]. Examples of groups satisfying (BC′) include all one-relator groups [132, Theorem 4.5], and fundamental groups of closed three-manifolds [105, Theorem 3.5]. See also [132, Theorem 4.6] for the case of Haken three-manifolds.

Using known permanence results, the class of groups for which $C^*(\Gamma)$ satisfies the UCT can be pushed a bit further: for example, if $\Gamma = \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ then $C^*(\Gamma)$ satisfies the UCT[26], even though Γ has relative property (T) with respect to the infinite subgroup $\mathbb{Z}^2$.

As far as we are aware, there are no examples of groups for which $C^*(\Gamma)$ is known to not satisfy the UCT. However, it seems very likely that the UCT fails for $C^*(\Gamma)$ for some (and maybe all) infinite property (T) groups: the computations Skandalis carries out in [129, Section 4] to show that the *reduced* group C^*-algebras of certain hyperbolic property (T) groups do not satisfy the UCT strongly suggest the same happens for the corresponding *maximal* group C^*-algebras (see also [129] and [72, Sections 6.2 and 6.3]), but we were not able to make this precise.

We do not know of any (separable) C^*-algebras that fail the approximate K-homology UCT of Definition 3.22, which is all we actually need in this paper.

Remark 5.7. We now say some more about the RFD assumption from Definition 1.12. Note first that a group being RFD can be expressed in purely group theoretic terms: Γ being RFD is equivalent to the set of all (equivalence classes of) finite-dimensional unitary representations of Γ being dense in the Fell topology on the unitary dual.

Recall that a group is *maximally almost periodic* (MAP) if it has a separating family of finite-dimensional unitary representations: for finitely generated groups, this is equivalent to residual finiteness by Mal'cev's theorem (see for example [28, Theorem 6.4.12]), but not in general as shown by $\Gamma = \mathbb{Q}$. If Γ is RFD, it is clearly MAP. For amenable groups the converse is true [21, Proposition 1], but not in general: for example Bekka shows in [21] that many arithmetic groups such as $SL(n,\mathbb{Z})$ with $n > 2$ are not RFD; they are, however, residually finite by Mal'cev's theorem again.

[25] There seems to be a slight gap in the statement given there: as well as (BC′), one also needs the C^*-algebra $\mathcal{A}$ appearing in the definition of (BC′) to be locally induced from C^*-algebras satisfying the UCT. This is satisfied in all the examples we give.

[26] This does not appear to be in the literature, but essentially the same proof as for $C^*(SL(2,\mathbb{Z}))$ works, once one has observed that the crossed product of $C^*(\mathbb{Z}^2)$ by any finite group is type I.

In general, the RFD property is known to be preserved under direct products with MAP amenable groups (this is straightforward), free products [62], certain free products amalgamated over finite subgroups [95, Theorem 2], and certain free products amalgamated over central subgroups and HNN extensions [127]. A large class of one-relator groups are shown to be RFD in [69, Theorem 3.11].

On the other hand, for F_2 the free group on two generators, whether $F_2 \times F_2$ is RFD is equivalent to Kirchberg's QWEP conjecture: this equivalence is implicit in [85, Proof of Proposition 8.1], and made explicit in [112, Proposition 3.19]. Thanks to the recent negative solution of the QWEP conjecture (see [50] for a survey), $F_2 \times F_2$ is therefore not RFD. The difficulty of the problem for $F_2 \times F_2$ suggests the difficulty of determining the RFD property in general.

A stronger property than being RFD that is both interesting in its own right and a good source of examples is *property FD* of Lubotzky and Shalom [102]. This says that the family of all unitary group representations that factor through finite quotients of the group separates points of $C^*(\Gamma)$. This is known for several interesting examples such as fundamental groups of surfaces [102, Theorem 2.8 (2)], free by cyclic groups (i.e. groups of the form $F \rtimes \mathbb{Z}$ for F a finitely generated free group) [102, Theorem 2.8 (1)]. It is also a consequence of [102, Theorem 2.8 (3)] that fundamental groups of closed hyperbolic three-manifolds that fiber over the circle have property FD; thanks to the solution of the virtually Haken conjecture [1] and the fact that property FD passes to finite index supergroups [102, Corollary 2.5], this implies that all fundamental groups of closed hyperbolic three manifolds satisfy property FD.

Finally, let us give some remarks about other techniques that seem relevant and that one might guess could be used to strengthen Theorem 5.3. We would be very interested to see progress along these lines (but, needless to say, are not currently able to make any).

Remark 5.8. Inspired by work of Kubota [89, Section 5.2], it is tempting to work not with $C^*(\Gamma)$, but with the C^*-algebra $C^*_{fd}(\Gamma)$ one gets by completing $\mathbb{C}[\Gamma]$ in the direct sum of all finite-dimensional representations. This has the advantage that $C^*_{fd}(\Gamma)$ is automatically RFD. However, there are two important disadvantages.

First, even if Γ is a-T-menable, it is not clear that the UCT holds for $C^*_{fd}(\Gamma)$. Indeed, this would hold for any completion of the group algebra that comes from a so-called *correspondence functor* by [29, Theorem 6.6]. However, any completion of $\mathbb{C}[\Gamma]$ coming from a correspondence functor has the property that its dual space is an ideal in the dual $C^*(\Gamma)^*$ of $C^*(\Gamma)$ [29, Corollary 5.7]. The trivial representation is the unit of $C^*(\Gamma)^*$; as the trivial representation is an element of $C^*_{fd}(\Gamma)^*$, we see that $C^*_{fd}(\Gamma)$ comes from a correspondence functor if and only if it equals $C^*(\Gamma)$.

The second problem with $C^*_{fd}(\Gamma)$ is that it is not clear which ucp quasi-representations extend to it, unlike for $C^*(\Gamma)$ where this is automatic by Proposition 2.2 part (ii) above.

Remark 5.9. Following the work of Dadarlat [41] it is also very tempting to work not with $C^*(\Gamma)$ but with the group convolution algebra $\ell^1(\Gamma)$. This would have three big advantages. First, it is always RFD when Γ is MAP. Second, (X, ϵ)-$*$-homomorphisms always extend to it, so one does not have to worry about ucp maps and analogues of Proposition 2.2; thus the LLP is likely not at all relevant. Third, one might be able to work directly with the Bost assembly map [3, Section 1.2] in place of the maximal Baum-Connes assembly map for explicit computations (see Section 6 below). The Bost assembly map is known to be an isomorphism in quite a lot more generality than the maximal Baum-Connes assembly map thanks to deep work of Lafforgue [90].

The ingredient that is missing in this case is an analogue of Theorem 3.25. Indeed, it is crucial for our methods that homomorphisms between K-theory groups can be realized by elements of bivariant KK-theory, and that these elements of KK-theory can be modeled by almost multiplicative maps. Lafforgue's Banach algebra KK-theory [91] might help here, but this does not seem obvious.

6 Index theory and the Baum-Connes conjecture

For a group Γ, the assembly maps of Baum-Connes [16, Section 3] and Kasparov [80, Section 6] relate the K-homology of the classifying space $B\Gamma$ to the K-theory of the group C^*-algebra $C^*(\Gamma)$. In this section, we use the Baum-Connes assembly map to compute the map

$$\pi_* \in \mathrm{Hom}_{\Lambda_0}(\underline{K}_0(C^*(\Gamma)), \underline{K}_0(\mathbb{C}))$$

of Example 3.14 induced by a finite-dimensional unitary representation $\pi : \Gamma \to M_n(\mathbb{C})$. These computations will be used to apply our abstract main result - Theorem 5.3 above - to concrete examples.

To summarize the rest of this section, Subsection 6.1 gives properties of the map $\pi_* : K_0(C^*(\Gamma)) \to K_0(\mathbb{C})$ induced by a unitary representation $\pi : \Gamma \to M_n(\mathbb{C})$ on K-theory with integer coefficients; this is all folklore and we are just summarizing what we need for the reader's convenience. Subsection 6.2 computes the maps $\pi_* : K_0(C^*(\Gamma); \mathbb{Z}/n) \to K_0(\mathbb{C}; \mathbb{Z}/n)$ induced by the representation π on K-theory with finite coefficients in terms of the *relative eta invariants* of Atyiah, Patodi, and Singer [12], at least in some special cases. As far as we know, this is new, but we suspect at least some of it is known to experts.

6.1 The Baum-Connes-Kasparov assembly map

In this subsection we recall background on the Baum-Connes-Kasparov assembly map, use it to compute the map

$$\pi_* \in \mathrm{Hom}(K_0(C^*(\Gamma)), K_0(\mathbb{C}))$$

arising from a finite-dimensional unitary representation, and collect some related facts. The computations we need are all part of the folklore of the subject, as will be clear from the references below.

Let X be a possibly infinite CW complex. We define the *representable K-homology* of X by

$$RK_*(X) := \lim_{\to} K_*(Y) \tag{34}$$

where the direct limit is taken over all finite subcomplexes $Y \subseteq X$ and $K_*(Y) = KK(C(Y), \mathbb{C})$ is analytic K-homology. We will mainly apply this to the case when X is the classifying space $B\Gamma$ of a group.

Assume now that Γ is a countable discrete group equipped with a CW complex model for the classifying space $B\Gamma$. The *Baum-Connes-Kasparov (BCK) assembly map* (see [16, Sections 3 and 7] and [80, Definition 6.2]) is a graded group homomorphism

$$\mu : RK_*(B\Gamma) \to K_*(C^*(\Gamma)). \tag{35}$$

To be explicit: we define the map μ above to be the same as the map β of [80, Definition 6.2], i.e. it is given by Kasparov product with the explicit element[27]

$$[\beta_\Gamma] \in RKK(B\Gamma; \mathbb{C}, C^*(\Gamma)) \tag{36}$$

defined there (up to a small technicality with direct limits).

In the next section, we will also need the BCK assembly map with finite coefficients, and record this now. For $n \geqslant 2$, we define *representable K-homology with finite coefficients* for a CW complex X to be

$$RK_*(X; \mathbb{Z}/n) := \lim_{\to} KK_*(C(Y), O_{n+1})$$

where the limit is taken over finite subcomplexes Y of X, O_{n+1} is as usual the Cuntz algebra, and KK is Kasparov's bivariant KK-theory [80]. The *BCK assembly map with coefficients in $\mathbb{Z}/n$*

$$\mu : RK_*(B\Gamma; \mathbb{Z}/n) \to K_*(C^*(\Gamma); \mathbb{Z}/n) \tag{37}$$

is again defined by Kasparov product with the element in line (36) above.

Lemma 6.1. *Assume that the BCK assembly map of line (35) is an isomorphism. Then for each $n \geqslant 2$ the BCK assembly map with finite coefficients of line (37) is also an isomorphism.*

[27] Sometimes called the *Miscenko bundle*.

Proof. We focus on the even case of the assembly map $RK_0(B\Gamma; \mathbb{Z}/n) \to K_0(C^*(\Gamma); \mathbb{Z}/n)$; the odd case is essentially the same. For an abelian group G, let ${}^nG$ and ${}_nG$ denote respectively the cokernel and kernel of the multiplication by n map $\times n : G \to G$; equivalently, ${}^nG = G \otimes \mathbb{Z}/n$ and ${}_nG$ is the n-torsion subgroup of G. The universal coefficient exact sequence[28] in K-theory and K-homology (see [124, Proposition 1.8] for K-theory, and the comments in [124, Section 5] for K-homology) give a diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}^nRK_0(B\Gamma) & \longrightarrow & RK_0(B\Gamma;\mathbb{Z}/n) & \longrightarrow & {}_nRK_1(B\Gamma) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & {}^nK_0(C^*(\Gamma)) & \longrightarrow & K_0(C^*(\Gamma);\mathbb{Z}/n) & \longrightarrow & {}_nK_1(C^*(\Gamma)) & \longrightarrow & 0 \end{array}$$

where the bottom and top lines are the exact universal coefficient sequences (more precisely, for the top line we take the exact universal coefficient sequence for each finite subcomplex Y, then take a direct limit), and the vertical maps are those induced by the BCK assembly map. All the maps are induced by Kasparov product with appropriate elements of KK-theory (see for example [47, Section 1.2] for the horizontal maps and [80, Definition 6.2] for the BCK assembly maps) whence the diagram commutes by associativity of the Kasparov product. The result follows from the five lemma. $\square$

For our main theorems, we will need to assume that the BCK assembly map is an isomorphism. Let us say a little more about this assumption, via its relation with the Baum-Connes assembly map.

Remark 6.2. Recall that the reduced group C^*-algebra $C_r^*(\Gamma)$ is the completion of the group algebra in its image under the left regular representation on $\ell^2(\Gamma)$. The *Baum-Connes assembly map* [16, Line (3.15)] for a discrete group Γ is a group homomorphism

$$\mu : K_*^\Gamma(\underline{E}\Gamma) \to K_*(C_r^*(\Gamma)) \tag{38}$$

from the equivariant topological K-homology of the classifiying space $\underline{E}\Gamma$ for proper actions to the K-theory of the reduced group C^*-algebra. If Γ is torsion-free $\underline{E}\Gamma$ equals the universal cover $E\Gamma$ of the classifying space $B\Gamma$, and this reduces to a map

$$\mu : RK_*(B\Gamma) \to K_*(C_r^*(\Gamma)).$$

[28]Potentially confusingly, there are (at least) two distinct short exact sequences that go by the name "universal coefficient exact sequence" in K-theory; the one we are using here should not be confused with the UCT assumption from [123] that we have used at various other points in this paper.

that agrees with the map of line (35) as long as one knows that the canonical quotient map

$$C^*(\Gamma) \to C^*_r(\Gamma) \tag{39}$$

induces an isomorphism on K-theory.

We will mainly be interested in cases where the BCK assembly map of line (35) is an isomorphism. This is obstructed in two important ways connected to the definition of the Baum-Connes assembly map in line (38).

First, the BCK assembly map is never surjective if Γ has non-trivial torsion elements. Indeed, let

$$\mathrm{tr} : \mathbb{C}[\Gamma] \to \mathbb{C}, \quad \sum_{g \in \Gamma} a_g g \mapsto a_e$$

be the canonical trace on the complex group algebra. This extends to a trace on $C^*(\Gamma)$, and this trace induces a map $\mathrm{tr}_* : K_0(C^*(\Gamma)) \to \mathbb{R}$. Surjectivity of the maximal BCK assembly map implies that tr_* is integer-valued: this is essentially Atiyah's covering index theorem [10] (see for example [140, Section 10.1] for a textbook treatment of this integrality result, and [103] for a further-reaching analysis). However, if $g \in \Gamma$ is a non-trivial element of order n for some $n \geqslant 2$, then $p := \frac{1}{n}\sum_{k=0}^{n-1} g^k$ defines a projection in the group ring such that $\mathrm{tr}_*[p] = 1/n$. This is the main reason for preferring the equivariant K-homology of $\underline{E}\Gamma$ (see [16, Sections 1 and 2]) to the K-homology of $B\Gamma$ in the definition of the Baum-Connes assembly map as in line (38).

Second, property (T) obstructs surjectivity of the BCK assembly map due to the existence of *Kazhdan projections* $p \in C^*(\Gamma)$ whose K-theory classes cannot be in the image of the assembly map; the essential point is again Atiyah's covering index theorem as explained in [71, Section 5]. This is one of the motivations for using the reduced group C^*-algebra $C^*_r(G)$ on the right hand side of the Baum-Connes conjecture as in line (38).

Nonetheless, the BCK assembly map in line (35) is known to be an isomorphism in many interesting cases. In all the cases we know this to hold, it happens for the conjunction of three reasons: the Baum-Connes assembly map as in line (38) is known to be an isomorphism; the group Γ is torsion-free, whence $K^\Gamma_*(\underline{E}\Gamma)$ agrees with $RK_*(B\Gamma)$; and the group Γ is K*-amenable* in the sense of Cuntz [38, Theorem 2.1 and Definition 2.2], which implies that the canonical map in line (39) induces an isomorphism on K-theory.

The most prominent case where these three conditions hold is that of torsion-free a-T-menable groups [73, Theorems 1.1 and 1.2], and in particular for torsion-free amenable groups. Another important case is torsion-free one-relator groups as in [20, Theorems 3 and 4]. A useful general criterion for the conditions above to hold is when Γ is torsion-free and satisfies Tu's *condition (BC′)* [132, Definition 2.5 and Lemma 2.6]. Examples of groups satisfying (BC′) include all one-relator groups again [132, Theorem 4.5], and fundamental groups of closed three-manifolds [105, Theorem 3.5]. See also [20, Theorem A] and [132, Theorem 4.6] for the case of Haken three-manifolds.

Following[29] [8, Section 4] need a slightly ad hoc[30], but technically convenient, definition.

Definition 6.3. Let X be a CW complex. We define the *limit K-theory* of X to be the inverse limit $\mathcal{K}^*(X) := \varprojlim K^*(Y)$ over all finite subcomplexes Y of X

Note that with this definition the usual pairings[31] $K^*(Y) \otimes K_*(Y) \to \mathbb{Z}$ between K-theory and K-homology of a compact space Y induce a canonical pairing

$$\mathcal{K}^*(X) \otimes RK_*(X) \to \mathbb{Z} \tag{40}$$

defined for any CW complex X. Analogously, the Kasparov product

$$KK_i(\mathbb{C}, C(Y)) \otimes KK_i(C(Y), O_{n+1}) \to KK(\mathbb{C}, O_{n+1}) = K_0(O_{n+1}) = \mathbb{Z}/n \tag{41}$$

for $i \in \{0, 1\}$ gives rise to a pairing

$$\mathcal{K}^i(X) \otimes RK_i(X; \mathbb{Z}/n) \to \mathbb{Z}/n. \tag{42}$$

For the next result, note that a vector bundle E over a CW complex X defines a class in $\mathcal{K}^0(X)$ in a canonical way: indeed, the restrictions of E to all finite subcomplexes define an element of the inverse system. Recall moreover that if $\pi : \Gamma \to M_m(\mathbb{C})$ is a unitary representation and $E\Gamma$ is the universal cover of the classifying space $B\Gamma$ then the associated *flat bundle* is defined by

$$E_\pi := (E\Gamma \times \mathbb{C}^m)/\Gamma \tag{43}$$

where Γ acts diagonally via deck transformations on $E\Gamma$, and via π on $\mathbb{C}^m$. Then the projection map $E_\pi \to E\Gamma/\Gamma = B\Gamma$ makes E_π into a rank m vector bundle over $B\Gamma$.

[29] More precisely, in [8, Section 4], Atiyah defines $\mathcal{K}^*(X)$ to be the inverse limit $\varprojlim K^*(X^{(n)})$, where $X^{(n)}$ is the n-skeleton of X; however, Atiyah is working with CW complexes with finitely many cells in each dimension; the definition we give here seems the correct generalization.

[30] We say "ad-hoc" as it is perhaps more common to use *representable K-theory* as in [80, Definition 2.19] or [126, Section 5]. The inverse limit version described here is more convenient for us; the two should be related by a Steenrod-Milnor sequence, but we did not check this.

[31] See for example [74, Section 7.2], or treat it as a special case of the Kasparov product [80, Theorem 2.14].

Lemma 6.4. *Let Γ be a discrete group, let $\pi : \Gamma \to M_m(\mathbb{C})$ be a unitary representation, and let E_π be the corresponding flat bundle over $B\Gamma$. Let $[E_\pi] \in \mathcal{K}^0(B\Gamma)$ be the corresponding class, and let $p_\pi : RK_0(B\Gamma) \to \mathbb{Z}$ be the operation of pairing with E_π as in line (40) above. Then the diagram*

$$\begin{array}{ccc} RK_0(B\Gamma) & \xrightarrow{\mu} & K_0(C^*(\Gamma)) \\ \downarrow{\scriptstyle p_\pi} & & \downarrow{\scriptstyle \pi_*} \\ \mathbb{Z} & = & \mathbb{Z} \end{array}$$

commutes. Similarly, the for any $n \geqslant 2$, the diagram

$$\begin{array}{ccc} RK_0(B\Gamma; \mathbb{Z}/n) & \xrightarrow{\mu} & K_0(C^*(\Gamma); \mathbb{Z}/n) \\ \downarrow{\scriptstyle p_\pi} & & \downarrow{\scriptstyle \pi_*} \\ \mathbb{Z}/n & = & \mathbb{Z}/n \end{array}$$

Proof. This can be established in the same way as [117, Lemma 4.2] (in the special case that the auxiliary space X appearing there is a point): we leave the details to the reader. □

For the next result, let $\tau : C^*(\Gamma) \to \mathbb{C}$ be the trivial representation. As τ splits the unit inclusion $\mathbb{C} \to C^*(\Gamma)$, there is a canonical splitting

$$K_0(C^*(\Gamma)) = \mathbb{Z}[1] \oplus \widetilde{K}_0(C^*(\Gamma)) \tag{44}$$

where $[1] \in K_0(C^*(\Gamma))$ is the class of the unit, and $\widetilde{K}_0(C^*(\Gamma))$ is the kernel of $\tau_* : K_0(C^*(\Gamma)) \to \mathbb{Z}$. Let also $\widetilde{RK}_0(B\Gamma)$ be the kernel of the map $RK_0(B\Gamma) \to \mathbb{Z}$ induced by collapsing $B\Gamma$ to a single point, and let $[pt] \in RK_0(B\Gamma)$ be the class induced by the inclusion of any point in $B\Gamma$, which also gives rise to a direct sum decomposition

$$RK_0(B\Gamma) = \mathbb{Z}[pt] \oplus \widetilde{RK}_0(B\Gamma) \tag{45}$$

Corollary 6.5. *Let Γ be a discrete group. With notation as in the paragraph above, the BCK assembly map of line (35) splits as a direct sum*

$$\widetilde{\mu} \oplus id_{\mathbb{Z}} : \widetilde{RK}_0(B\Gamma) \oplus \mathbb{Z}[pt] \to \widetilde{K}_0(C^*(\Gamma)) \oplus \mathbb{Z}[1].$$

Proof. Lemma 6.4 implies that the diagram below

$$\begin{array}{ccc} \text{Kernel}(p_\tau) & \xrightarrow{\mu} & \widetilde{K}_0(C^*(\Gamma)) \\ \downarrow & & \downarrow \\ RK_0(B\Gamma) & \xrightarrow{\mu} & K_0(C^*(\Gamma)) \\ \downarrow{\scriptstyle p_\tau} & & \downarrow{\scriptstyle \tau_*} \\ \mathbb{Z} & = & \mathbb{Z} \end{array}$$

commutes. The map p_τ is the operation of pairing with the trivial line bundle: this agrees with the map $RK_0(B\Gamma) \to \mathbb{Z}$ defined by collapsing $B\Gamma$ to a point, so $\text{Kernel}(p_\tau) = \widetilde{RK}_0(B\Gamma)$. The lower two vertical maps p_τ and τ_* are split by the inclusion of a point, and the inclusion of the unit respectively, and the corresponding diagram still commutes by naturality of the assembly map (the bottom horizontal arrow can be thought of as the BCK assembly map for the trivial group). The result follows from these observations. ☐

The next lemma is again part of the folklore of the subject: see for example [84, Theorem 2.4.3] or [42, Proposition 3.2] for a proof.

Lemma 6.6. *Let Γ be a discrete group, and let μ be the BCK assembly map of line (35). Let $\pi : \Gamma \to M_m(\mathbb{C})$ be a finite-dimensional unitary representation. Then with respect to the splitting in line (45), $\pi_*(\mu(\widetilde{RK}_0(B\Gamma))) = 0$ and $\pi_*(\mu[pt]) = m$.* ☐

Remark 6.7. If $\mu : RK_*(B\Gamma) \to K_*(C^*(\Gamma))$ is an isomorphism, then combining Lemmas 6.5 and 6.6, we see that if $\pi : \Gamma \to M_m(\mathbb{C})$ is a unitary representation then

$$\pi_*(\widetilde{K_0}(C^*(\Gamma))) = 0 \quad \text{and} \quad \pi_*([1]) = m. \tag{46}$$

It is at first (at least to the author) surprising that something with such an elementary statement needs to be proved using fairy heavy machinery like the BCK assembly map. However, the formulas in line (46) will fail for any non-trivial representation of a property (T) group; as such, it seems there cannot be a genuinely elementary proof.

6.2 Finite coefficients and eta invariants

In Subsection 6.1, we used the BCK assembly map of line (35) to compute the map

$$\pi_* \in \text{Hom}(K_0(C^*(\Gamma)), K_0(\mathbb{C}))$$

induced by a unitary representation $\pi : \Gamma \to M_m(\mathbb{C})$. In this section, we aim to compute the maps

$$\pi_* \in \text{Hom}(K_0(C^*(\Gamma); \mathbb{Z}/n), K_0(\mathbb{C}; \mathbb{Z}/n))$$

on K-theory with finite coefficients induced by π. This is more subtle: it turns out to be more convenient to compute the difference

$$\tau_* - \pi_* : \text{Hom}(K_0(C^*(\Gamma); \mathbb{Z}/n), K_0(\mathbb{C}; \mathbb{Z}/n)) \tag{47}$$

where τ is a trivial representation of the same dimension as π, and we do this in terms of the *relative eta invariant* of Atiyah, Patodi, and Singer [12].

We will follow the approach of Higson and Roe to relative eta invariants from [75]. For this, we need to recall the Baum-Douglas geometric model of K-homology [17]; more specifically, we will use the variant of this discussed by Higson and Roe in [75, Section 3], which is in turn based on work of Keswani [83, Section 2.2]. Keswani's version of the geometric model of K-homology is shown to be equivalent to the original Baum-Douglas version in [83, Section 2.3]. We briefly recall the cycles for odd geometric K-homology group $K_1(X)$ of a compact metric space X in this model; the even group is defined analogously, but we will not need this.

Definition 6.8. An *odd K-cycle* for a compact metric space X is a triple (M, S, f) where:

(i) M is a smooth, closed, odd-dimensional, orientable, Riemannian manifold (it need not be connected, and the connected components need not have the same dimension as long as all are odd-dimensional);

(ii) S is a Dirac bundle over M in the sense of [75, Definition 3.1] (the precise details of what this means will not be important to us);

(iii) $f : M \to X$ is a continuous map.

The odd geometric K-homology group $K_1(X)$ is then defined to consist of K-cycles modulo the equivalence relation defined in [75, Definition 3.11] (again, the exact details are not important to us). The groups $K_1(X)$ are covariantly functorial: if $g : X \to Y$ is a continuous map then $g_* : K_1(X) \to K_1(Y)$ is the map induced on K-cycles by

$$(M, S, f) \mapsto (M, S, g \circ f). \tag{48}$$

We will also need Dirac operators associated to K-cycles. Given an odd K-cycle (M, S, f) one can associate a *Dirac operator*

$$D = D^{M,S} \tag{49}$$

as in [75, Definition 3.5]: this is a first order elliptic operator acting on smooth sections of the bundle S. The Dirac operator D is not uniquely determined – it depends on a choice of 'compatible connection' as in [119, Definition 3.4] – but the choices involved will not be important for us.

There is a well-defined map from geometric K-homology to analytic K-homology defined on K-cycles by

$$(M, S, f) \mapsto f_*[D], \tag{50}$$

where D is a choice of Dirac operator as in line (49), and $[D]$ is the class it defines in analytic K-homology (see for example [74, Chapter 10]). This map defines an isomorphism between geometric and analytic K-homology for finite CW complexes: see [77, Section 4], [118, Chapter 4], or [18] for a proof. Note that if X is a possibly infinite CW complex and $RK_1(X)$

is defined via a direct limit as in line (34), then classes in $RK_1(X)$ can still be described as triples (M, S, f) with $f : M \to X$ a continuous map as in Definition 6.8 above: indeed, as $f : M \to X$ is continuous and M is compact, f takes image in a finite subcomplex of X.

Let now M be a closed, odd-dimensional, Riemannian manifold, equipped with a Dirac bundle S as in Definition 6.8 and a choice of Dirac operator D on S. In their first paper on the subject [11], Atiyah, Patodi, and Singer use spectral data to associate a real number $\eta(D)$ to D called the *eta invariant*. Very roughly, $\eta(D)$ is the difference between the number of positive eigenvalues of D and the number of negative eigenvalues; however, both numbers are typically infinite so this does not make literal sense, and $\eta(D)$ is actually defined using a sort of 'zeta function regularization'.

The invariant $\eta(D)$ is quite delicate: it depends on D, not only on the underlying topological data from the manifold M and bundle S. In the second paper [12] of their series, Atiyah, Patodi, and Singer use $\eta(D)$ to define a more robust *relative eta invariant* as follows. Assume that M is equipped with a homomorphism $\pi_1(M) \to \Gamma$, and let $\pi : \Gamma \to M_m(\mathbb{C})$ be a finite dimensional unitary representation. This data defines a rank m flat bundle E_π over M analogously to line (43) above, and we write D_π for the Dirac operator D twisted[32] by this bundle. On the other hand, let D_m denote D twisted by the rank m trivial bundle $\mathbb{C}^m$ on M. Let k_π (respectively, k_m) denote the dimension of the kernel of D_π (respectively, of D_m) acting on smooth sections of the associated bundle tensored by E_π (respectively, by $\mathbb{C}^m$). Then the *relative eta invariant*[33] is defined in [12, Line (3.2)] by

$$\rho_\pi(D) := \frac{k_\pi + \eta(D_\pi)}{2} - \frac{k_m + \eta(D_m)}{2} \in \mathbb{R}/\mathbb{Z}. \tag{51}$$

The reason for considering $\rho_\pi(D)$ as an element of $\mathbb{R}/\mathbb{Z}$ rather than $\mathbb{R}$ is that is has stronger invariance properties this way. From a modern point of view, the invariance properties of ρ_π are well-summarized by the following theorem, which is [75, Theorem 6.1]; the reader might also usefully compare this to the material in [2, Section 6] which gives an approach based on von-Neumann algebras rather than the Baum-Douglas geometric model for K-homology.

Theorem 6.9 (Higson-Roe)**.** *Let (M, S, f) be a K-cycle as in Definition 6.8, and $D = D^{M,S}$ a choice of associated Dirac operator as in line (49). Equip $\pi_1(M)$ with the homomorphism $f_* : \pi_1(M) \to \pi_1(B\Gamma) = \Gamma$*

[32]See for example [93, page 139] or [119, Example 3.24] for a Dirac operator twisted by (or equivalently, 'with coefficients in') a vector bundle.

[33]We warn the reader that we use the notation "ρ" by analogy with [75, Definition 2.3], but Atiyah-Patodi-Singer instead use "$\widetilde{\xi}_\pi(0)$"; what we call "ρ_π" is not the same as what Atiyah-Patodi-Singer call "ρ_π" in [12, Theorem 2.4].

induced by f, and let $\pi : \Gamma \to M_m(\mathbb{C})$ be a finite dimensional unitary representation. Then the assignment

$$(M, S, f) \mapsto \rho_\pi(D)$$

descends to a well-defined homomorphism

$$\rho_\pi : RK_1(B\Gamma) \to \mathbb{R}/\mathbb{Z}. \qquad \square$$

Our first main goal in this section is to relate the map in line (47) to the relative eta homomorphism ρ_π of Theorem 6.9. We need some more K-theoretic ingredients that we now describe.

Recall (see for example [124, Proposition 1.6] or [47, line (1.6)]) that that for a C^*-algebra A and $n \geqslant 2$ there is a *Bockstein six-term exact sequence*

$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\times n} & K_0(A) & \xrightarrow{\rho_n} & K_0(A; \mathbb{Z}/n) \\ \beta_n \uparrow & & & & \downarrow \beta_n \\ K_1(A; \mathbb{Z}/n) & \xleftarrow[\rho_n]{} & K_1(A) & \xleftarrow[\times n]{} & K_1(A) \end{array} \quad . \qquad (52)$$

The maps are all induced by Kasparov products with particular elements of KK-groups, although we will not need precise descriptions. There is an analogous six-term exact sequence in (representable) K-homology

$$\begin{array}{ccccc} RK_0(B\Gamma) & \xrightarrow{\times n} & RK_0(B\Gamma) & \xrightarrow{\rho_n} & RK_0(B\Gamma; \mathbb{Z}/n) \\ \beta_n \uparrow & & & & \downarrow \beta_n \\ RK_1(B\Gamma; \mathbb{Z}/n) & \xleftarrow[\rho_n]{} & RK_1(B\Gamma) & \xleftarrow[\times n]{} & RK_1(B\Gamma) \end{array} \qquad (53)$$

defined using Kasparov product with the same elements mentioned above, and then taking a direct limit over finite subcomplexes of $B\Gamma$ (here we use that direct limits preserve exactness). The maps β_n in line (52) or (53) are usually called *Bockstein homomorphisms.*

Here is the first main result of this section. We suspect it may be know to some experts, but are not aware of it appearing in the literature before.

Theorem 6.10. *Let Γ be a discrete group. Let $\pi : \Gamma \to M_m(\mathbb{C})$ be a unitary representation of Γ such that the restriction of π to any finitely generated subgroup factors through a finite quotient. Let $\tau = \tau^{(m)} : \Gamma \to M_m(\mathbb{C})$ be the m-dimensional trivial representation. Let $n \geqslant 2$. Then the following diagram commutes*

$$\begin{array}{ccccc} RK_0(B\Gamma; \mathbb{Z}/n) & \xrightarrow{\mu} & K_0(C^*(\Gamma); \mathbb{Z}/n) & \xrightarrow{\tau_* - \pi_*} & \mathbb{Z}/n \\ \downarrow \beta_n & & & & \downarrow \\ RK_1(B\Gamma) & & \xrightarrow{\qquad\rho_\pi\qquad} & & \mathbb{R}/\mathbb{Z} \end{array}$$

where: the top right horizontal arrow is the map of line (47); the top left horizontal arrow is the BCK assembly map with finite coefficients of line (37); the left vertical arrow is the Bockstein homomorphism appearing in the exact sequence of line (53); the right vertical arrow sends the canonical generator $1 \in \mathbb{Z}/n$ to $1/n$; and the bottom horizontal arrow ρ_π is the relative eta invariant homomorphism of Theorem 6.9.

This is almost implict in the papers of Atiyah, Patodi, and Singer [12, 13]. However, those authors do not work with K-homology (which was not available in a convenient model at that time), but rather with K-theory of tangent bundles. The two are related by Kasparov's Poincaré duality: see [81, Section 4] for the version we will use. The proof will consist largely of combining these ingredients, but we must first recall some background.

The first collection of facts we need is based on [12, Section 5]. Let A be a (unital) C^*-algebra, and for each $n, m \geqslant 1$, let

$$\kappa_{nm,n} : K_*(A; \mathbb{Z}/n) \to K_*(A; \mathbb{Z}/(nm)) \tag{54}$$

be the natural transformation induced by any unital $*$-homomorphism $O_{n+1} \to M_m(O_{nm+1})$ inducing the canonical injection $\mathbb{Z}/n \to \mathbb{Z}/(nm)$ defined by sending 1 to m; such a homomorphism exists by the Kirchberg-Phillips classification theory as for example in [64, Theorem A]. Let $\mathcal{N}$ denote the set $\{n \in \mathbb{N} \mid n \geqslant 2\}$ equipped with the partial order where $n \leqslant m$ if n divides m. Then the collection

$$(K_*(A; \mathbb{Z}/n))_{n \in \mathcal{N}} \tag{55}$$

is a directed system with the maps $\kappa_{nm,n}$ as connecting maps. The following definitions are based on [12, Section 5].

Definition 6.11. Let A be a C^*-algebra. We define the *K-theory of A with $\mathbb{Q}/\mathbb{Z}$ coefficients* as the direct limit

$$K_*(A; \mathbb{Q}/\mathbb{Z}) := \lim_{\rightarrow} K(A; \mathbb{Z}/n)$$

over the directed set of line (55). If X is a possibly infinite CW complex, we define

$$\mathcal{K}^*(X; \mathbb{Q}/\mathbb{Z}) := \lim_{\leftarrow} K^*(Y; \mathbb{Q}/\mathbb{Z})$$

where the inverse limit is taken over all finite subcomplexes Y of X analogously to Definition 6.3.

Analogously to line (42) above for any compact metric space Y and each $n \geqslant 2$, the Kasparov product induces a pairing

$$KK_1(\mathbb{C}, C(Y) \otimes O_{n+1}) \otimes KK_1(C(Y), \mathbb{C}) \to KK(\mathbb{C}, O_{n+1})$$

or in other words

$$K^1(Y; \mathbb{Z}/n) \otimes K_1(Y) \to K_0(O_{n+1}) = \mathbb{Z}/n.$$

Taking the direct limit over the directed system in line (55) then gives a pairing

$$K^1(Y;\mathbb{Q}/\mathbb{Z}) \otimes K_1(Y) \to \lim_{n\in\mathcal{N}} K_0(O_{n+1}) = \mathbb{Q}/\mathbb{Z}. \tag{56}$$

Finally, if X is a general CW complex, these pairings induce a pairing

$$\mathcal{K}^1(X;\mathbb{Q}/\mathbb{Z}) \otimes RK_1(X) \to \mathbb{Q}/\mathbb{Z} \tag{57}$$

by taking limits over finite CW subcomplexes.

Now, the Bockstein exact sequences of line (52) for n and nm fit into diagrams

$$\begin{array}{ccccccccc}
\cdots \xrightarrow{\times n} & K_1(A) & \xrightarrow{\rho_n} & K_1(A;\mathbb{Z}/n) & \xrightarrow{\beta_n} & K_0(A) & \xrightarrow{\times n} & K_0(A) & \xrightarrow{\rho_n} \cdots \\
& \downarrow{\scriptstyle \times m} & & \downarrow{\scriptstyle \kappa_{nm,n}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \times m} & \\
\cdots \xrightarrow{\times nm} & K_1(A) & \xrightarrow{\rho_{nm}} & K_1(A;\mathbb{Z}/(nm)) & \xrightarrow{\beta_{nm}} & K_0(A) & \xrightarrow{\times nm} & K_0(A) & \xrightarrow{\rho_{nm}} \cdots
\end{array} \tag{58}$$

(the top and bottom lines are 'unrolled' versions of the Bockstein six-term sequences from line (52)); these commute by the identities in [124, Proposition 2.1], plus the essential uniqueness of the natural transformations involved as in [31, Lemma A.4]. Taking the direct limit of the six-term exact sequences in line (58) over the directed set $\mathcal{N}$ defined just above line (55) gives a six-term exact sequence

$$\begin{array}{ccccc}
K_0(A) & \xrightarrow{\iota} & K_0(A)\otimes\mathbb{Q} & \xrightarrow{\rho} & K_0(A;\mathbb{Q}/\mathbb{Z}) \\
\uparrow{\scriptstyle \beta} & & & & \downarrow{\scriptstyle \beta} \\
K_1(A;\mathbb{Q}/\mathbb{Z}) & \xleftarrow[\rho]{} & K_1(A)\otimes\mathbb{Q} & \xleftarrow[\iota]{} & K_1(A)
\end{array} \tag{59}$$

where the arrows labeled ι are induced by the canonical inclusion $\mathbb{Z}\to\mathbb{Q}$, and the arrows labeled ρ and β are the direct limits of the arrows ρ_n and β_n respectively. Moreover, if $A = C(Y)$ is the continuous functions on a connected compact Hausdorff space, then passing to reduced K^0 groups $\widetilde{K}^0$ (and noting that the reduced K^1 groups are the same as the usual K^1 groups) the sequence

$$\begin{array}{ccccc}
\widetilde{K}^0(Y) & \xrightarrow{\iota} & \widetilde{K}^0(Y)\otimes\mathbb{Q} & \xrightarrow{\rho} & \widetilde{K}^0(Y;\mathbb{Q}/\mathbb{Z}) \\
\uparrow{\scriptstyle \beta} & & & & \downarrow{\scriptstyle \beta} \\
K^1(Y;\mathbb{Q}/\mathbb{Z}) & \xleftarrow[\rho]{} & K^1(Y)\otimes\mathbb{Q} & \xleftarrow[\iota]{} & K^1(Y)
\end{array} \tag{60}$$

is a direct summand of the sequence in line (59), and in particular still exact.

Finally, if X is a possibly infinite connected CW complex, we write

$$\beta : \mathcal{K}^1(X;\mathbb{Q}/\mathbb{Z}) \to \widetilde{\mathcal{K}}^0(X). \tag{61}$$

for the map induced by taking the inverse limits of the left hand vertical maps in line (60) as Y ranges over finite subcomplexes of X (this makes sense by naturality of β). Here we let $\widetilde{\mathcal{K}}^0(X)$ be the kernel of the map $\mathcal{K}^0(X) \to \mathbb{Z}$ induced by the inclusion of a point.

The next lemma is implicit in [12, Section 5]; we provide a proof for the reader's convenience. Recall first that if Λ is a finite group then there is a CW model for its classifying space that is finite in each dimension – for example, the infinite join construction of the classifying space from [107, Section 3] has this property – but if Λ is not trivial then there is no finite model for $B\Lambda$ (see for example [25, Corollary VIII.2.5]).

Lemma 6.12. *Let Λ be a finite group and let $B\Lambda$ be a CW complex model for its classifying space that is finite in each dimension. Then the map β of line (61) is an isomorphism for $X = B\Lambda$.*

Proof. Let $B\Lambda^{(n)}$ denote the n-skeleton of $B\Lambda$; as this is finite for all n, $\mathcal{K}^0(B\Lambda)$ is the inverse limit of the inverse system $(K^0(B\Lambda^{(n)}))_{n=1}^{\infty}$. Similarly, $\mathcal{K}^1(B\Lambda;\mathbb{Q}/\mathbb{Z})$ is the inverse limit of the system $(K^1(B\Lambda^{(n)};\mathbb{Q}/\mathbb{Z}))_{n=1}^{\infty}$.

As $B\Lambda$ has trivial rational cohomology[34], considering cellular cohomology shows that $\widetilde{H}^k(B\Lambda^{(n)};\mathbb{Q})$ is zero if $k \neq n$. As the Chern character is a rational isomorphism, $\widetilde{K}^0(B\Lambda^{(n)}) \otimes \mathbb{Q} = 0$ for n odd, and $K^1(B\Lambda^{(n)}) \otimes \mathbb{Q} = 0$ for n even. Now, consider the short exact sequence for n odd

$$0 \to \frac{K^1(B\Lambda^{(n)}) \otimes \mathbb{Q}}{\iota(K^1(B\Lambda^{(n)}))} \xrightarrow{\rho} K^1(B\Lambda^{(n)};\mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} \widetilde{K}^0(B\Lambda^{(n)}) \to 0 \qquad (62)$$

induced from line (60) and the above observations. As $K^1(B\Lambda^{(n)}) \otimes \mathbb{Q} = 0$ for n even, the connecting maps for the inverse system

$$\left(\frac{K^1(B\Lambda^{(n)}) \otimes \mathbb{Q}}{\iota(K^1(B\Lambda^{(n)}))}\right)_{n \text{ odd}}$$

are all zero. In particular, this inverse system satisfies the Mittag-Leffler condition so the inverse limit of the short exact sequence in line (62) is still exact (see for example [8, Pages 31-2] or [136, Section 3.5]), giving a short exact sequence

$$0 \to K \xrightarrow{\rho} \mathcal{K}^1(B\Lambda;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta} \widetilde{\mathcal{K}}^0(B\Lambda) \to 0.$$

The group K on the left is an inverse limit of a system with zero connecting maps, so is trivial, and we are done. □

We need one more lemma before the proof of Theorem 6.10.

[34] This is well-known, but we were unable to find a good reference. One can for example show this purely algebraically by showing that $\mathbb{Q}$ with the trivial Λ-action is a flat $\mathbb{Z}\Lambda$-module.

Lemma 6.13. *Let Γ be a countable group, let $y \in \mathcal{K}^1(B\Gamma; \mathbb{Q}/\mathbb{Z})$, and let $n \geqslant 2$. Then the diagram below commutes*

$$\begin{array}{ccc} RK_0(B\Gamma;\mathbb{Z}/n) & \xrightarrow{p_{\beta(y)}} & \mathbb{Z}/n \\ \downarrow{\scriptstyle \beta_n} & & \downarrow \\ RK_1(B\Gamma) & \xrightarrow{p_y} & \mathbb{Q}/\mathbb{Z} \end{array}$$

where: the left hand map β_n is the right vertical map from line (53); $\beta(y)$ is the image of y under the map β of line (61) and $p_{\beta(y)}$ is the operation of pairing with it as in line (42); p_y is the operation of pairing with y as in line (57); and the right hand vertical arrow is the canonical inclusion determined by $1 \mapsto 1/n$.

Proof. Thanks to the definitions of the pairings in lines (42) and (57), it suffices to show that if Y is a finite CW subcomplex of $B\Gamma$ and z is the image of y under the canonical map $\mathcal{K}^1(B\Gamma; \mathbb{Q}/\mathbb{Z}) \to K^1(Y; \mathbb{Q}/\mathbb{Z})$, then the following diagram commutes

$$\begin{array}{ccc} KK(C(Y), O_{n+1}) & \xrightarrow{p_{\beta(z)}} & KK(\mathbb{C}, O_{n+1}) = \mathbb{Z}/n \\ \downarrow{\scriptstyle \beta_n} & & \downarrow \\ KK_1(C(Y), \mathbb{C}) & \xrightarrow{p_z} & \lim_m KK(\mathbb{C}, O_{m+1}) = \mathbb{Q}/\mathbb{Z} \end{array}$$

where: β is the left hand vertical arrow in line (59) for $A = C(Y)$ and $p_{\beta(z)}$ is the result of pairing with $\beta(z) \in KK_1(\mathbb{C}, C(Y))$ as in line (41); p_z is the result of pairing with $z \in \lim_m KK(\mathbb{C}, C(Y) \otimes O_{m+1})$ as in line (56); and the two vertical maps are defined as before. Let now $m \in \mathbb{N}$ be such that n divides m, and such that z is in the image of an element z_m under the map $KK(\mathbb{C}, C(Y) \otimes O_{m+1}) \to \lim_m KK(\mathbb{C}, C(Y) \otimes O_{m+1})$. Then it suffices to show that the diagram below commutes

$$\begin{array}{ccc} KK(C(Y), O_{n+1}) & \xrightarrow{p_{\beta_m(z_m)}} & KK(\mathbb{C}, O_{n+1}) = \mathbb{Z}/n \\ \downarrow{\scriptstyle \beta_n} & & \downarrow{\scriptstyle \kappa_{m,n}} \\ KK_1(C(Y), \mathbb{C}) & \xrightarrow{p_{z_m}} & KK(\mathbb{C}, O_{m+1}) = \mathbb{Z}/m \end{array} \tag{63}$$

where $\kappa_{m,n} \in KK(O_{n+1}, O_{m+1})$ is as in line (54). Let $b_m \in KK_1(O_{m+1}, \mathbb{C})$ be the element inducing the map β_m, and similarly for b_n. Then the mage of $x \in KK(C(Y), O_{n+1})$ under the right-down composition in line (63) is given by

$$((z_m \cdot b_m) \cdot x) \cdot \kappa_{m,n}$$

while the down-right composition is given by

$$z_m \cdot (x \cdot b_n);$$

using associativity of the Kasparov product, we therefore must show that

$$b_m \cdot x \cdot \kappa_{m,n} = x \cdot b_n$$

in $KK_1(O_{m+1} \otimes C(Y), O_{m+1})$. Moreover, $b_m \cdot x = x \cdot b_m$, so it suffices to show that $b_m \cdot \kappa_{m,n} = b_n$ in $KK_1(O_{n+1}, \mathbb{C})$. It follows from [124, Proposition 2.1] that $b_m \cdot \kappa_{m,n}$ and b_n induce the same natural transformations on K-theory with finite coefficients, from which the identity $b_m \cdot \kappa_{m,n} = b_n$ follows (compare for example [123, Remark 8.8]). ☐

We are now ready for the proof of Theorem 6.10.

Proof of Theorem 6.10. The diagram appearing in Theorem 6.10 is continuous as one takes direct limits over groups. Thus we may assume that Γ is finitely generated, and that π factors through a finite quotient of Γ.

Let E_π be the flat bundle over $B\Gamma$ corresponding to π as in line (43) above, and let $\mathbb{C}^m$ be the corresponding trivial bundle of the same rank. Let

$$p_\pi : RK_0(B\Gamma; \mathbb{Z}/n) \to \mathbb{Z}/n, \quad x \mapsto ([E_\pi] - [\mathbb{C}^m]) \cdot x$$

be the operation of pairing with $[E_\pi] - [\mathbb{C}^m] \in \widetilde{\mathcal{K}}^0(B\Gamma)$ as in line (42) above. Then the diagram

$$\begin{array}{ccc} RK_0(B\Gamma; \mathbb{Z}/n) & \xrightarrow{\mu} & K_0(C^*(\Gamma); \mathbb{Z}/n) \\ \downarrow{\scriptstyle p_\pi} & & \downarrow{\scriptstyle \pi_* - \tau_*} \\ \mathbb{Z}/n & = & \mathbb{Z}/n \end{array}$$

commutes by Lemma 6.4. It thus suffices to show that

$$\begin{array}{ccc} RK_0(B\Gamma; \mathbb{Z}/n) & \xrightarrow{p_\pi} & \mathbb{Z}/n \\ \downarrow{\scriptstyle \beta_n} & & \downarrow \\ RK_1(B\Gamma) & \xrightarrow{\rho_\pi} & \mathbb{R}/\mathbb{Z} \end{array} \tag{64}$$

commutes.

Let $q : \Gamma \to \Lambda$ be a finite quotient of Γ through which π factors. Let $[E_\pi^\Lambda] - [\mathbb{C}^m] \in \widetilde{\mathcal{K}}^0(B\Lambda)$ be the corresponding class where E_π^Λ is the flat bundle associated to π (considered as a representation of Λ) as in line (43). Let $q : B\Gamma \to B\Lambda$ also denote the induced map on classifying spaces. Then the diagram

$$\begin{array}{ccc} \widetilde{\mathcal{K}}^0(B\Lambda) & \xrightarrow{q^*} & \widetilde{\mathcal{K}}^0(B\Gamma) \\ \uparrow{\scriptstyle \beta} & & \uparrow{\scriptstyle \beta} \\ \mathcal{K}^1(B\Lambda; \mathbb{Q}/\mathbb{Z}) & \xrightarrow{q^*} & \mathcal{K}^1(B\Gamma; \mathbb{Q}/\mathbb{Z}) \end{array} \tag{65}$$

commutes by naturality of the map β from line (61) above. The left hand map β in line (65) above is an isomorphism by Lemma 6.12, so we may define $y \in \mathcal{K}^1(B\Gamma; \mathbb{Q}/\mathbb{Z})$ by

$$y := q^*(\beta^{-1}([E_\pi^\Lambda] - [\mathbb{C}^m])) \in \mathcal{K}^1(B\Gamma; \mathbb{Q}/\mathbb{Z})$$

(compare [12, page 431]). Then by commutativity of the diagram in line (65), $\beta(y) = [E^\pi] - [\mathbb{C}^m] \in \widetilde{\mathcal{K}}^0(B\Gamma)$. Hence we have an equality of maps

$$p_{\beta(y)} = p_\pi : RK_0(B\Gamma; \mathbb{Z}/n) \to \mathbb{Z}/n. \tag{66}$$

where the map $p_{\beta(y)}$ is as in Lemma 6.13. As Lemma 6.13 gives us the commutative diagram

$$\begin{array}{ccc} RK_0(B\Gamma; \mathbb{Z}/n) & \xrightarrow{p_{\beta(y)}} & \mathbb{Z}/n \\ \downarrow{\scriptstyle \beta_n} & & \downarrow \\ RK_1(B\Gamma) & \xrightarrow{p_y} & \mathbb{Q}/\mathbb{Z} \end{array}$$

and as we have the identity in line (66), to show that the diagram in line (64) commutes (recall this is our goal!) if suffices to show an identity of maps

$$\rho_\pi = p_y : RK_1(B\Gamma) \to \mathbb{Q}/\mathbb{Z}. \tag{67}$$

This is essentially the Atiyah-Patodi-Singer index theorem for flat bundles, as we explain in the remainder of the proof.

Indeed, let $z \in RK_1(B\Gamma)$ be an arbitrary class represented by a K-cycle (M, S, f) as in Definition 6.8. Let $[D] \in K_1(M)$ be the class of an associated Dirac operator in analytic K-homology, and let $[\sigma_D]$ be the symbol class of D in $K^1(TM)$. Then on [13, page 87], Atiyah-Patodi-Singer construct an *analytical index map* $\text{ind}_\pi : K^1(TM) \to \mathbb{R}/\mathbb{Z}$ such that the equality

$$\text{ind}_\pi([\sigma_D]) = \rho_\pi(z) \tag{68}$$

holds by definition (compare Theorem 6.9 above). On the other hand on [13, page 87] again, Atiyah-Patodi-Singer define a *topological index map* $\text{Ind}_\pi : K^1(TM) \to \mathbb{R}/\mathbb{Z}$ by the formula

$$\text{Ind}_\pi([\sigma_D]) := \text{Ind}(f^*(y) \cdot_{TM} [\sigma_D]) \tag{69}$$

where: the product $\cdot_{TM}$ on the right hand side is the module action of $K^1(M; \mathbb{Q}/\mathbb{Z})$ on $K^1(TM)$

$$K^1(M; \mathbb{Q}/\mathbb{Z}) \otimes K^1(TM) \to K^0(TM)$$

(a variation incorporating suspensions and coefficients of the module action of $K^0(M)$ on $K^0(TM)$ defined on [14, page 491]); and Ind is the *Atiyah-Singer topological index map* [14, Section 3]

$$\text{Ind} : K^0(TM; \mathbb{Q}/\mathbb{Z}) \to K^0(pt; \mathbb{Q}/\mathbb{Z}) = \mathbb{Q}/\mathbb{Z}$$

adapted for $\mathbb{Q}/\mathbb{Z}$ coefficients. Using [13, Theorem 5.3] (compare also [13, Remark 2 on page 87]), the left hand sides of lines (68) and (69) agree, and so

$$\rho_\pi(z) = \operatorname{Ind}(f^*(y) \cdot_{TM} [\widetilde{\sigma_D}]). \tag{70}$$

To establish the formula in line (67), we follow Kasparov's approach to index theory as exposited in [81, Sections 2-4].

Now, analogously to Kasparov [81, page 1318], we may consider the symbol σ_D of D as defining a class in $KK_1(C(M), C_0(TM))$[35], which we denote ${}_M[\sigma_D]$. In the language of the Kasparov product, we have

$$f^*(y) \cdot_{TM} [\sigma_D] = f^*(y) \cdot {}_M[\sigma_D].$$

Thus we may rewrite line (69) as

$$\operatorname{Ind}_\pi([\sigma_D]) = \operatorname{Ind}(f^*(y) \cdot {}_M[\sigma_D]). \tag{71}$$

Let $[\mathcal{D}_M] \in K_0(TM) = KK(C_0(TM), \mathbb{C})$ denote Kasparov's *Dolbeault* K*-homology class* as in [81, Definition 2.8]. Then we have *Kasparov's Poincaré duality* (see [81, Theorem 4.2])

$${}_M[\sigma_D] \cdot [\mathcal{D}_M] = [D] \in K_1(M) \tag{72}$$

and (still by [81, Theorem 4.2], or more precisely, a slight variant with coefficients) that for any w in

$$K^0(TM; \mathbb{Q}/\mathbb{Z}) = \lim_n KK(\mathbb{C}, C_0(TM) \otimes O_{n+1})$$

we have that

$$\operatorname{Ind}(w) = [w] \cdot [\mathcal{D}_M]. \tag{73}$$

Hence if we write $[f] \in \lim_Y KK(C(Y), C(M))$ for the KK-class associated to f (the limit is taken over all finite subcomplexes Y of $B\Gamma$), then starting with lines (70) and (71), we have

$$\begin{aligned} \rho_\pi(z) &= \operatorname{Ind}(f^*(y) \cdot {}_M[\sigma_D]) \\ &= ((y \cdot [f]) \cdot {}_M[\sigma_D]) \cdot [\mathcal{D}_M] \end{aligned}$$

where we have used that $f^*(y) = y \cdot [f]$ (by definition) and line (73). Using associativity of the Kasparov product and Poincaré duality as in line (72), we thus get the first equality in the chain below

$$\rho_\pi(z) = y \cdot [f] \cdot [D] = y \cdot f_*[D] = y \cdot z = p_y(z).$$

[35] Kasparov just calls this class $[\sigma_D]$, and considers it as an element of a slightly more complicated group $\mathcal{R}K_1(M; C_0(TM))$, where the latter group would be denoted $\mathcal{R}KK_1(M; C(M), C_0(TM))$ in [80, 2.19] (compare [81, page 1306]). We only need the image of Kasparov's class under the forgetful map $\mathcal{R}K_1(M; C(M), C_0(TM)) \to KK_1(C(M), C_0(TM))$, so just work there for simplicity.

The other equalities are justified by: the second equality is by definition of $f_*[D]$; the third follows as $f_*[D]$ is the analytic K-homology class associated to the K-cycle (M, S, f) underlying z (compare line (50) above); and the last follows by definition of p_y (see Lemma 6.13 above). This is the desired identity from line (67), so we are done. ☐

For our concrete examples, we will need the following explicit computation. Again, it is essentially due to Atiyah, Patodi, and Singer.

Proposition 6.14. *Let Γ be a discrete group. Let $[S^1, \mathbb{C}, f] \in RK_1(B\Gamma)$ be the class associated to a continuous map $f : S^1 \to B\Gamma$. Let $\pi : \Gamma \to \mathbb{C}$ be a one-dimensional unitary representation. Let $f_* : \mathbb{Z} = \pi_1(S^1) \to \Gamma$ be the map induced on fundamental groups by f, and assume that $\pi \circ f_*$ takes $1 \in \mathbb{Z}$ (identified with the class of a loop of winding number one) to $e^{2\pi i q}$ for some $q \in (0, 1)$. Then*

$$\rho_\pi(\mu([S^1, \mathbb{C}, f])) = -q \in \mathbb{R}/\mathbb{Z}.$$

Proof. The relative eta invariant map of Theorem 6.9 is natural, as follows directly from the description of functoriality for K-cycles in line (48). Hence the diagram below commutes

$$\begin{array}{ccc} K_1(S^1) & \xrightarrow{\rho_{\pi\circ f_*}} & \mathbb{R}/\mathbb{Z} \\ \downarrow f_* & & \| \\ K_1(B\Gamma) & \xrightarrow{\rho_\pi} & \mathbb{R}/\mathbb{Z} \end{array}.$$

It thus suffices to show that if $\sigma : \pi_1(S^1) = \mathbb{Z} \to \mathbb{C}$ is the one-dimensional unitary representation taking 1 to $e^{2\pi i q}$, and

$$\rho_\sigma : K_1(B\mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$$

is the homomorphism of Theorem 6.9, then (having identified S^1 with $B\mathbb{Z}$), $\rho_\sigma[S^1, \mathbb{C}, \mathrm{id}] = -q$. Now, a canonical choice of Dirac operator associated to the K-cycle $(S^1, \mathbb{C}, \mathrm{id})$ is $D = i\frac{d}{dx}$ acting on the trivial line bundle (here we treat S^1 as $\mathbb{R}/\mathbb{Z}$ and x as the usual variable). With notation as in line (51) above, we thus need to show that

$$\frac{k_\sigma + \eta(D_\sigma)}{2} - \frac{k_1 + \eta(D_1)}{2} = -q$$

The necessary direct computations for this are contained in [12, page 411]. Indeed, for $0 < q < 1$, [12, page 411] shows that D_σ has trivial kernel, and the kernel of D_1 is spanned by the constant functions; hence $k_\sigma = 0$ and $k_1 = 1$. On the other hand, it is also computed in [12, page 411] that $\eta(D_\sigma) = 1 - 2q$ and that $\eta(D_1) = 0$. The result follows. ☐

The next theorem puts together our work in Section 6; it was motivated partly by comments of Marius Dadarlat.

Theorem 6.15. *Let Γ be a discrete group, for which the BCK assembly map μ of line (35) is an isomorphism. Assume moreover that any torsion class in $RK_1(B\Gamma)$ can be represented by a K-cycle of the form $(S^1, \mathbb{C}, f)$ for some continuous map $f : S^1 \to B\Gamma$.*

Then with notation as in Definition 3.13, Example 3.14, and Definition 3.20, for any K-datum P for $\underline{K}_0(C^*(\Gamma))$ there exists a finite subset Q of the group $\widetilde{K_0}(C^*(\Gamma))$ of line (44) with the following property.

Assume that

$$\alpha \in Hom_{\Lambda_0}(\underline{K}_0(C^*(\Gamma)), \underline{K}_0(\mathbb{C}))$$

is such that the induced map $\alpha_1 : K_0(C^*(\Gamma)) \to K_0(\mathbb{C})$ satisfies that $\alpha_1[1] \geqslant 0$ and α_1 vanishes on Q. Then there exists a character $\sigma : \Gamma \to \mathbb{C}$ such that if $\tau : \Gamma \to \mathbb{C}$ is the trivial representation then

$$\alpha + \sigma_* = (\alpha[1] + 1)\tau_* \tag{74}$$

as elements of the set $Hom_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$ of Definition 3.20.

Proof. For an abelian group G, let ${}^nG$ and ${}_nG$ denote respectively the cokernel and kernel of the multiplication by n map $\times n : G \to G$ (equivalently, ${}^nG = G \otimes \mathbb{Z}/n$ and ${}_nG$ is the n-torsion subgroup of G). As in the proof of Lemma 6.1, we have the short exact universal coefficient theorem sequence[36]

$$0 \to {}^nK_0(A) \xrightarrow{\rho_n} K_0(A; \mathbb{Z}/n) \xrightarrow{\beta_n} {}_nK_1(A) \to 0.$$

As in [31, Lemma A.5], this sequence admits a splitting

$$s_0 : {}_nK_1(A) \to K_0(A; \mathbb{Z}/n)$$

that is compatible with the operations making up the category Λ_0[37] from Definition 3.13.

Specialize now to $A = C^*(\Gamma)$, and consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & {}^nK_0(C^*(\Gamma)) & \xrightarrow{\rho_n} & K_0(C^*(\Gamma); \mathbb{Z}/n) & \xrightarrow{\beta_n} & {}_nK_1(C^*(\Gamma)) & \longrightarrow & 0\,. \\ & & \downarrow{\scriptstyle \tau_*} & & \downarrow{\scriptstyle \tau_*} & & & & \\ & & \mathbb{Z}/n & = \!= \!= & \mathbb{Z}/n & & & & \end{array} \tag{75}$$

The vertical maps are split by the unit inclusion, so we have a direct sum decomposition

$$K_0(C^*(\Gamma); \mathbb{Z}/n) \cong \operatorname{Ker}(\tau_*) \oplus \mathbb{Z}/n$$

[36] It is the result of 'unsplicing' the long-exact sequence of line (52).

[37] It is not possible to choose such a splitting that is natural for $*$-homomorphisms between C^*-algebras.

and in particular, there is an idempotent map $p : K_0(C^*(\Gamma); \mathbb{Z}/n) \to K_0(C^*(\Gamma); \mathbb{Z}/n)$ with image $\mathrm{Ker}(\tau_*)$. Define

$$s : {}_nK_1(C^*(\Gamma)) \to K_0(C^*(\Gamma); \mathbb{Z}/n), \quad s := p \circ s_0. \tag{76}$$

Using the diagram in line (75), one sees that this is still a splitting for $\beta_n : K_0(C^*(\Gamma); \mathbb{Z}/n) \to {}_nK_1(C^*(\Gamma))$, and thus we get a direct sum decomposition

$$K_0(C^*(\Gamma); \mathbb{Z}/n) \cong {}^nK_0(C^*(\Gamma)) \oplus {}_nK_1(C^*(\Gamma)) \tag{77}$$

that is compatible with the operations in Λ_0 as n varies, and also so that the summand ${}_nK_1(C^*(\Gamma))$ is contained in the kernel of τ_*. Using our assumption that the BCK assembly map is an isomorphism (which implies that the BCK assembly map with any $\mathbb{Z}/n$ coefficients is an isomorphism by Lemma 6.1), we pull this back to a decomposition

$$RK_0(B\Gamma; \mathbb{Z}/n) \cong {}^nRK_0(B\Gamma) \oplus {}_nRK_1(B\Gamma) \tag{78}$$

that is compatible with Λ_0 and with the BCK assembly map μ.

Now, let P be a K-datum for $\underline{K}_0(C^*(\Gamma))$ as in the statement. Write P_1 for the part of P in $K_0(C^*(\Gamma))$, and P_n for the part in $K_0(C^*(\Gamma); \mathbb{Z}/n)$ for $n \geqslant 2$; all but finitely many of the P_n will be empty. For each $n \geqslant 2$, write $P_{n,0}$ for the projection of P_n onto the direct summand ${}^nK_0(C^*(\Gamma))$ in line (77). Let Q be a finite subset of the group $\widetilde{K_0}(C^*(\Gamma))$ of line (44) that is large enough so that $Q + \mathbb{Z}[1]$ contains P_1, and so that for each n, the image of $Q + \mathbb{Z}[1]$ under the map

$$\rho_n : K_0(C^*(\Gamma)) \to K_0(C^*(\Gamma); \mathbb{Z}/n)$$

of line (52) contains $P_{n,0}$. We claim that Q has the property in the statement.

Assume then that we are given

$$\alpha \in \mathrm{Hom}_{\Lambda_0}(\underline{K}_0(C^*(\Gamma)), \underline{K}_0(\mathbb{C}))$$

such that $\alpha_1[1] \geqslant 0$, and so that $\alpha_1(Q) = 0$. We need to define σ with the properties in the statement.

To do this, for each $n \geqslant 2$, the definition of Hom_{Λ_0} (see Definition 3.13 above) gives maps

$$\alpha_n : K_0(C^*(\Gamma); \mathbb{Z}/n) \to K_0(\mathbb{C}; \mathbb{Z}/n) = \mathbb{Z}/n.$$

Abusing notation slightly we also write $\alpha_n : {}_nK_1(C^*(\Gamma)) \to \mathbb{Z}/n$ for the restriction of α_n to the direct summand in line (77). For each $m \geqslant 2$

compatibility of α, and of the direct sums decompositions in line (77), with the operations in Λ_0 implies that we have a commutative diagram

$$\begin{array}{ccc} {}_nK_1(C^*(\Gamma)) & \xrightarrow{\alpha_n} & \mathbb{Z}/n \\ \downarrow & & \downarrow \\ {}_{nm}K_1(C^*(\Gamma)) & \xrightarrow{\alpha_{nm}} & \mathbb{Z}/(nm) \end{array}$$

where the vertical maps are the canonical inclusions (compare the commutative diagram in [31, Line (A.28)]). Taking the direct limit of these diagrams over all n in the directed set $\mathcal{N}$ defined just above line (55) therefore gives us a map

$$\alpha_{tor} : \text{Tor}(K_1(C^*(\Gamma)) \to \mathbb{Q}/\mathbb{Z},$$

where $\text{Tor}(\cdot)$ denotes the torsion subgroup of an abelian group. Let Γ_{ab} be the abelianization of Γ, and define σ_0 to the composition

$$\text{Tor}(\Gamma_{ab}) \to \text{Tor}(K_1(C^*(\Gamma)) \xrightarrow{\alpha_{tor}} \mathbb{Q}/\mathbb{Z}$$

where the first map is induced by the map $\Gamma \to K_1(C^*(\Gamma))$ sending group elements to the canonical unitaries. Then σ_0 extends to a map $\sigma_1 : \Gamma_{ab} \to \mathbb{Q}/\mathbb{Z}$ by injectivity (i.e. divisibility - see for example [136, Corollary 2.3.2]) of the abelian group $\mathbb{Q}/\mathbb{Z}$. We define $\sigma : \Gamma \to \mathbb{C}$ to be the complex conjugate of the composition

$$\Gamma \to \Gamma_{ab} \xrightarrow{\sigma_1} \mathbb{Q}/\mathbb{Z} \to \mathbb{C}$$

where the first map is the canonical quotient, and the last map is $x \mapsto e^{2\pi ix}$. Note that the restriction of σ to any finitely generated subgroup of Γ takes image in a finitely generated, torsion, abelian group, and thus the image is finite. We claim that the equation in line (74) holds on P, which will complete the proof.

Indeed, by choice of Q and assumption that α_1 vanishes on it, α_1 agrees with $\alpha[1]\tau_*$ on $P_1 \subseteq K_0(C^*(\Gamma))$; moreover, σ_* agrees with τ_* on this summand by Lemma 6.6. Similarly, α_n agrees with $\alpha[1]\tau_*$ on the image $P_{n,0}$ of each P_n in ${}^nK_0(C^*(\Gamma))$ by compatibility of α with Λ_0, and σ_* agrees with τ_* on this set. Moreover, τ_* vanishes on the summand ${}_nK_1(C^*(\Gamma))$ in line (77) by choice of the splitting s from line (76). To complete the proof, it thus suffices to show that for each $n \geqslant 2$ the restriction of α_n to the summand ${}_nK_1(C^*(\Gamma))$ agrees with σ_*; it moreover suffices to show that it agrees with $\tau_* - \sigma_*$.

For this, we use commutativity of the diagram

$$\begin{array}{ccccc} RK_0(B\Gamma;\mathbb{Z}/n) & \xrightarrow{\mu} & K_0(C^*(\Gamma);\mathbb{Z}/n) & \xrightarrow{\tau_*-\sigma_*} & \mathbb{Z}/n \\ \downarrow{\scriptstyle \beta_n} & & & & \downarrow \\ RK_1(B\Gamma) & & \xrightarrow{\rho_\sigma} & & \mathbb{R}/\mathbb{Z} \end{array}$$

from Theorem 6.10. Using our assumption that the BCK assembly map μ is an isomorphism (which implies that it is also an isomorphism with all finite coefficients by Lemma 6.1), and also naturality of μ with respect to β_n, it thus suffices to show that we have an equality

$$\alpha_n \circ \mu = \rho_\sigma : {}_nRK_1(B\Gamma) \to \mathbb{R}/\mathbb{Z} \tag{79}$$

for each n (we use here that the map from $RK_0(B\Gamma; \mathbb{Z}/n)$ to the summand ${}_nRK_1(B\Gamma)$ as in line (78) is β_n).

Fix then $n \geqslant 2$, and let $z \in {}_nRK_1(B\Gamma)$ be an arbitrary class. Our assumption on torsion classes in $RK_1(B\Gamma)$ implies that $z = [S^1, \mathbb{C}, f]$ for some continuous map $f : S^1 \to B\Gamma$. Proposition 6.14 thus gives that

$$\rho_\sigma(z) = q$$

where $\sigma \circ f_*(1) = e^{-2\pi i q}$. On the other hand, by definition of σ,

$$\sigma(f_*(1)) = e^{-2\pi i \alpha_{tor}([f_*(1)])} = e^{-2\pi i \alpha_n [f_*(1)]}$$

where $[f_*(1)]$ is the class of $f_*(1)$ in $K_1(C^*(\Gamma))$. Hence to show that the identity in line (79) holds, it suffices to show that

$$[f_*(1)] = \mu[S^1, \mathbb{C}, f].$$

To see this note that the identification $B\mathbb{Z} = S^1$ and naturality of the assembly map gives us a commutative diagram

$$\begin{array}{ccc} RK_1(S^1) & \xrightarrow{\mu} & K_1(C^*(\mathbb{Z})) \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\ RK_1(B\Gamma) & \xrightarrow{\mu} & K_1(C^*(\Gamma)) \end{array} .$$

The two groups on the top line are isomorphic to $\mathbb{Z}$, and are generated by $[S^1, \mathbb{C}, \mathrm{id}]$ and $[1]$ respectively. The left hand vertical map takes $[S^1, \mathbb{C}, \mathrm{id}]$ to $[S^1, \mathbb{C}, f]$ (compare line (48) above), and the right hand vertical map takes $[1]$ to $[f_*(1)]$, so we are done. $\square$

7 Low-dimensional examples

In this section, we use the results of Sections 5 and 6 to do explicit computations for some low-dimensional examples.

The first result we need comes from Matthey's paper [104]. We warn the reader that Matthey uses "K_*" for what we call "RK_*" (see line (34) above).

Proposition 7.1. *For any connected CW complex X there are homomorphisms*

$$\beta_i^X : H_i(X) \to RK_{i \bmod 2}(X)$$

for $i = 0, 1, 2$ that are natural for continuous maps between CW-complexes.

The map β_0^X has the property that $\beta_0^X[pt]_H = [pt]_K$ where $[pt]_H \in H_0(X)$ (respectively, $[pt]_K \in RK_0(X)$) is the class coming from the inclusion of any point. The map β_1^X has the property that if $f : S^1 \to X$ describes an element of $\pi_1(X)$ and $[f] \in H_1(B\Gamma)$ is the image of this element under the Hurewicz map $\pi_1(X) \to H_1(X)$, then

$$\beta_1^X([f]) = [S^1, \mathbb{C}, f]$$

(here we use notation for K-cycles as in Definition 6.8).

Moreover, if X is a CW complex of dimension at most three, then the direct sum map

$$\beta_0^X \oplus \beta_2^X : H_0(X) \oplus H_2(X) \to RK_0(X)$$

is an isomorphism, and there is an isomorphism

$$\beta_{od}^X : H_1(X) \oplus H_3(X) \to RK_3(X).$$

that restricts to β_1^X on $H_1(X)$.

Proof. The existence and naturality of β_0^X, plus the fact that $\beta_0^X[pt]_H = [pt]_K$ is contained in [104, Proposition 2.1 (i)]. Existence and naturality of β_1^X and β_2^X is contained in [104, Proposition 3.2]. The explicit description of β_1^X in terms of loops is contained in [104, Proposition 3.4]. Finally, the isomorphism statement follows from [104, Proposition 2.1 (ii) and Proposition 3.3]. ☐

We next need to recall a result due to Dadarlat [44, Theorem 1.1]. To state it, we recall that $H_2(B\Gamma)$ identifies with the second group homology $H_2(\Gamma)$ (essentially by definition), and that if $\Gamma = \langle S \mid R \rangle$ is a (possibly infinite) presentation giving rise to a short exact sequence

$$\{e\} \to \langle\langle R \rangle\rangle \to F_S \to \Gamma \to \{e\}$$

(here F_S is the free group on S, and $\langle\langle R \rangle\rangle$ is the normal subgroup of F_S generated by R) then Hopf (see [25, Theorem 5.3]) showed that

$$H_2(\Gamma) = \frac{\langle\langle R \rangle\rangle \cap [F, F]}{[F, \langle\langle R \rangle\rangle]},$$

where $[A, B]$ denotes the subgroup generated by commutators $[a, b]$ with $a \in A$, $b \in B$ and A, B subgroups of some ambient group G. Following Dadarlat, an element c of $H_2(B\Gamma)$ can therefore be represented as a product

$$\prod_{i=1}^{g} [a_i, b_i]$$

where $a_i, b_i \in F_S$, and the image $\prod_{i=1}^g [\overline{a_i}, \overline{b_i}]$ in Γ is trivial. Let $\phi : \Gamma \to \mathcal{B}(\mathbb{C}^n)$ be a quasi-representation. Then as long as $\prod_{i=1}^g [\phi(\overline{a_i}), \phi(\overline{b_i}]$ is within distance one of the identity, the path

$$[0,1] \ni t \mapsto \det\Big((1-t) + t\prod_{i=1}^g [\phi(\overline{a_i}), \phi(\overline{b_i}]\Big) \tag{80}$$

passes only through $\mathbb{C}^\times := \{z \in \mathbb{C} \mid \neq 0\}$, and starts and ends at 1.

Definition 7.2. Let Γ be a countable group, let $c \in H_2(B\Gamma)$ be given by

$$\prod_{i=1}^g [a_i, b_i]$$

as above, and let $\phi : \Gamma \to M_n(\mathbb{C})_1$ be a quasi-representation such that

$$\Big\| \prod_{i=1}^g [\phi(\overline{a_i}), \phi(\overline{b_i})] - 1 \Big\| < 1.$$

We write $w(\phi, c)$ for the winding number of the path in line (80) above.

Dadarlat's theorem [44, Theorem 1.1] is more general than the statement we give below in that it works without the ucp assumption. We state it like this just to fit it more cleanly into our framework.

Theorem 7.3 (Dadarlat)**.** *Let Γ be a countable discrete group, and let $c \in H_2(B\Gamma)$. Then for any K-datum P as in Definition 3.20 that contains $\mu(\beta_2^{B\Gamma}(c))$ there exists a finite subset S of Γ and $\epsilon > 0$ such that (P, S, ϵ) is a $\underline{K}_0$-triple in the sense of Definition 4.1, such that $w(\phi, c)$ makes sense, and with the following property.*

If $\phi : \Gamma \to M_n(\mathbb{C})$ is a ucp (S, ϵ)-representation in the sense of Definition 1.3 and

$$\kappa_P^{(S,\epsilon)} : Hom_{\Lambda_0}(\underline{K}_0^\epsilon(S), \underline{K}_0(\mathbb{C})) \to Hom_{\Lambda_0}(P, \underline{K}_0(\mathbb{C}))$$

is as in Lemma 3.21 above, then

$$\kappa_P^{(S,\epsilon)}(\phi_*)(\mu(\beta_2^{B\Gamma}(c))) = w(c, \phi). \qquad \square$$

Here is our main general theorem for groups with low-dimensional classifying space.

Theorem 7.4. *Let Γ be a countable discrete group such that $C^*(\Gamma)$ is RFD and satisfies the approximate K-homology UCT of Definition 3.22. Assume that the BCK assembly map of line (35) is an isomorphism. Assume moreover that the classifying space $B\Gamma$ is realized as a CW complex of dimension at most three, and that $H_3(B\Gamma)$ is torsion free.*

Let $\mathcal{S}$ be a collection of finite symmetric subsets of Γ that is closed under finite unions, and with the property that $\bigcup_{S \in \mathcal{S}} S$ generates Γ. Then *for any $S \in \mathcal{S}$ and $\epsilon > 0$ there exist $T \in \mathcal{S}$, $\delta > 0$ and a finite subset C of $H_2(B\Gamma)$ with the following property.*

For any ucp (T, δ) representation $\phi : \Gamma \to M_n(\mathbb{C})$ the winding number invariants $w(\phi, c)$ make sense for $c \in C$. Moreover, if $w(\phi, c) = 0$ for all $c \in C$, then there are unitary representations $\theta : \Gamma \to M_k(\mathbb{C})$ and $\pi : \Gamma \to M_{n+k}(\mathbb{C})$ such that

$$\|\phi(s) \oplus \theta(s) - \pi(s)\| < \epsilon$$

for all $s \in S$.

If moreover Γ has the LLP of Definition 2.4 and if $\mathcal{S}$ is the collection of all finite symmetric subsets of Γ, then one can drop the assumption that ϕ is ucp.

Proof. Let (S, ϵ) be given, and let (P, T, δ) be as in the conclusion of Theorem 5.3 (in the non-LLP case). Let $Q \subseteq \widetilde{K}_0(C^*(\Gamma))$ be a finite set as in the conclusion of Theorem 6.15 for the K-datum P. Using Proposition 7.1, the assumption that the BCK assembly map is an isomorphism, and Corollary 6.5, there is a finite subset C of $H_2(B\Gamma)$ such that $\mu(\beta_2^{B\Gamma}(C)) = Q$. Expanding T and shrinking δ if necessary, we may assume that $w(c, \phi)$ is defined whenever $\phi : \Gamma \to M_n(\mathbb{C})$ is a ucp (T, δ)-representation.

Assume then that $\phi : \Gamma \to M_n(\mathbb{C})$ is a ucp (T, δ)-representation such that the winding number invariants $w(\phi, c)$ are zero for all $c \in C$. Then Theorem 7.3 implies that with $\kappa_P^{(S,\epsilon)}$ as in Lemma 3.21, we have that $\kappa_P^{(S,\epsilon)}(\phi_*(q)) = 0$ for all $q \in Q$. Note that Proposition 7.1 implies that $RK_1(B\Gamma) \cong H_1(B\Gamma) \oplus H_3(B\Gamma)$; as by assumption $H_3(B\Gamma)$ is torsion free, all torsion in $RK_1(B\Gamma)$ comes from $\beta_1^{B\Gamma}(H_1(B\Gamma))$, and therefore by Proposition 7.1 again, every torsion class in $RK_1(B\Gamma)$ is of the form $[S^1, \mathbb{C}, f]$ for some continuous map $f : S^1 \to B\Gamma$. Let $\alpha \in \mathrm{Hom}_{\Lambda_0}(\underline{K_0}(C^*(\Gamma)), \underline{K_0}(\mathbb{C}))$ be any element that equals $\kappa_P^{(S,\epsilon)}(\phi_*)$ in $\mathrm{Hom}_{\Lambda_0}(P, \underline{K_0}(\mathbb{C}))$, and note that $\alpha_1[1] = \phi_*[1] = n \geqslant 0$. Then by Theorem 6.15 there is a character $\sigma : \Gamma \to \mathbb{C}$ such that

$$\kappa_P^{(S,\epsilon)}(\phi_*) + \sigma_* = (n+1)\tau_*$$

as elements of $\mathrm{Hom}_{\Lambda_0}(P, \underline{K_0}(\mathbb{C}))$, where τ is the trivial representation. Write $\tau^{(n)}$ for the n-dimensional trivial representation.

We may now apply Theorem 5.3 to conclude that there exists a representation $\theta_0 : \Gamma \to M_{k_0}(\mathbb{C})$ and a unitary $u \in M_{n+k_0+1}(\mathbb{C})$ such that

$$\|u(\phi(s) \oplus \sigma(s) \oplus \theta_0(s))u^* - \tau^{(n+1)}(s) \oplus \theta_0(s)\| < \epsilon$$

for all $s \in S$. Setting $k = k_0 + 1$, $\theta = \sigma \oplus \theta_0$, and $\pi = u^*(\tau^{(n+1)} \oplus \theta_0)u$, we are done. ☐

7.1 Free-by-cyclic groups

In this subsection we discuss *free-by-cyclic* groups, i.e. groups of the form $F \rtimes_\phi \mathbb{Z}$ where F is a finitely generated free group and $\mathbb{Z}$ acts on F_n by an automorphism ϕ. This class of groups is large and well-studied. Note that it contains $\mathbb{Z}^2$ and the Klein bottle group $\mathbb{Z} \rtimes \mathbb{Z}$ as special cases, and that these (and the degenerate case $\Gamma = \mathbb{Z}$) are the only amenable examples.

The following result summarizes key facts about such groups. We believe these are all well-known, but do not know where to look them up.

Proposition 7.5. *Let $\Gamma = F \rtimes_\phi \mathbb{Z}$ be a free-by-cyclic group where F is a finitely generated free group, and ϕ is an automorphism of F. Then Γ is UCT, RFD, LLP, and the BCK assembly map for Γ is an isomorphism.*

Moreover, Γ admits a finite CW complex $B\Gamma$ of dimension two for its classifying space, and $H_2(B\Gamma)$ is a free-abelian group with rank equal to the multiplicity of the eigenvalue one for the map

$$\phi_* : H_1(F; \mathbb{C}) \to H_1(F; \mathbb{C})$$

induced by ϕ on homology with complex coefficients.

Proof. The group Γ is clearly torsion-free. It has the Haagerup property by [33, Example 6.1.6]. As discussed in Remarks 5.6 and 6.2 respectively, the Haagerup property implies that Γ is UCT and that the BCK assembly map is an isomorphism. The RFD and LLP results are covered by Remarks 5.7 and 5.5 respectively.

Let now BF be the classifying space for F, which we realize in the usual way as a wedge of n circles where n the rank of F. Let $\phi : BF \to BF$ be a map that induces ϕ on fundamental groups (such exists and is unique up to homotopy). We may assume that ϕ is cellular (see [70, Theorem 4.8]), whence the mapping torus defined by

$$M_\phi := (BF \times [0, 1])/((x, 0) \sim (1, \phi(x)))$$

can be given the structure of a finite, two-dimensional CW complex. It moreover admits a fibration $BF \to M_\phi \to S^1$, and is thus aspherical by the long exact sequence of homotopy groups for a fibration (see for example [70, Theorem 4.41]). The fundamental group $\pi_1(M_\phi)$ agrees with Γ by the Seifert-van Kampen theorem, and therefore M_ϕ is the requited model for $B\Gamma$. The statement on H_2 follows from the long exact sequence

$$0 \to H_2(M_\phi) \to H_1(BF) \xrightarrow{\mathrm{id}-\phi_*} H_1(BF) \to H_1(M_\phi) \to H_0(BF) \to 0$$

for the homology of the mapping torus as in [70, Example 2.48]. □

The next result follows directly from Proposition 7.5 and Theorem 7.4.

Theorem 7.6. *Let $\Gamma = F \rtimes_\phi \mathbb{Z}$ be a free-by cyclic group. Let n be the rank of F, and let $\phi_* : \mathbb{R}^n \to \mathbb{R}^n$ be the map induced by ϕ on the first homology group. Let m be the rank of one as an eigenvalue of ϕ_*, and let $c_1, ..., c_m$ be a corresponding basis for $H_2(B\Gamma)$. Then Γ is weakly stable, conditional on vanishing of the winding number invariants $w(c_i, \phi)$ for all $i \in \{1, ..., m\}$.*

In particular, if ϕ_* does not have one as as an eigenvalue, then Γ is weakly stable. □

We do not know if Γ as in Theorem 7.6 is (conditionally) stable. This seems an interesting question. It also seems plausible that there is a more elementary proof of Theorem 7.6, but this is not immediately obvious to us.

7.2 One relator groups

A finitely generated group Γ is called a *one relator group* if it admits a presentation of the form $\Gamma = \langle S \mid r \rangle$ with S finite and where r is a single word. Important examples include the fundamental group of an orientable surface of genus g

$$\Sigma_g := \left\langle a_1, ..., a_g, b_1, ..., b_g \mid \prod_{i=1}^{g} [a_i, b_i] \right\rangle, \tag{81}$$

the fundamental group of a non-orientable surface of genus g

$$N_g := \left\langle a_1, ..., a_{g+1} \mid \prod_{i=1}^{g+1} a_i^2 \right\rangle \tag{82}$$

and the Baumslag-Solitar groups

$$BS(m, n) := \langle a, b \mid ba^m b^{-1} = a^n \rangle. \tag{83}$$

All one-relator groups other than $BS(1, m)$ for some m, and (the degenerate case of) cyclic groups are non-amenable: see [32, page 338]. One-relator groups always satisfy the UCT as noted in Remark 5.6, but whether or not they are RFD or LLP seems to need to be handled on a case-by-case basis.

We need to recall some information about homology and classifying spaces of one relator groups: see for example [25, Section II.4, example 3] for more information on what follows. Let $\Gamma = \langle S \mid r \rangle$ be a one-relator group. Then Γ is torsion free if and only if r is not of the form u^n for some word u and $n \geqslant 2$. Moreover, if Γ is torsion-free then the presentation two-complex (i.e. the two-dimensional complex constructed for example in [70, Corollary 1.28]) is a $B\Gamma$. The homology of Γ is (therefore) given by $H_0(B\Gamma) = \mathbb{Z}$, $H_1(B\Gamma) = \Gamma_{ab}$ the abelianization of Γ, and

$$H_2(B\Gamma) = \begin{cases} \mathbb{Z} & r \in [\Gamma, \Gamma] \\ 0 & \text{otherwise} \end{cases}$$

(here $[\Gamma, \Gamma]$ is the commutator subgroup of Γ. All the higher homology vanishes. Thus we get the following result.

Theorem 7.7. *Let $\Gamma = \langle S \mid r \rangle$ be a one-relator group, and F_S be the free group on the generators of S. Assume moreover that Γ is torsion-free and RFD. Then the following hold.*

(i) If r is not in the commutator subgroup of F_S, then Γ is weakly ucp stable.

(ii) If r is in the commutator subgroup of S, let $w(r, \phi)$ be the associated winding number invariant of a quasi-representation ϕ as in Definition 7.2. Then Γ is weakly ucp stable, conditional on vanishing of $w(r, \phi)$.

Moreover, if Γ is LLP, then we may drop the word "ucp" from both parts of the above.

Proof. As commented in Remarks 5.6 and 6.2 respectively, Γ is UCT, and the BCK assembly map for Γ is an isomorphism. The result now follows from Theorem 7.4. □

Let us discuss some specific cases where the groups in question are known to be RFD in more detail.

Example 7.8. Perhaps the most interesting case where Theorem 7.7 applies is to fundamental groups of surfaces, i.e. the groups Σ_g and N_g from lines (81) and (82) above. As in Remark 5.7, [102, Theorem 2.8 (2)] shows that these are RFD. Hence Theorem 7.7 applies: this is Theorem 1.8 from the introduction. This in some sense gives a complement to the non-stability result of [56, Theorem 4.8], and the discussion of [41, Section 4].

It does not seem to be known whether fundamental groups of surfaces satisfy the LLP. This seems to be a very interesting question; we conjecture they do[38].

Example 7.9. Let us make some comments about the Baumslag-Solitar groups $BS(n, m)$ of line (83).

First, note that these groups are rarely RFD: in fact, they are not even residually finite unless $|m| = |n|$, or one of $|m|$ or $|n|$ equals one [106, Theorem C]. On the other hand, $BS(n, n)$ can be written as $\mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}$, where the amalgamation subgroup $\mathbb{Z}$ is included in $\mathbb{Z}^2 = \mathbb{Z} \oplus \mathbb{Z}$ as the first summand, and is included in the second summand as $n\mathbb{Z}$. Hence $BS(n, n)$ is RFD by [88, Theorem 1.1]. On the other hand, the subgroup of $BS(n, -n) = \langle a, b \mid ab^n a^{-1} b^n \rangle$ generated by a^2 and b has index two, and is isomorphic to $BS(n, n)$. Groups with finite index RFD subgroups

[38] This should probably be regarded as a folk conjecture, but I am not aware of it appearing in the literature.

are themselves RFD (the same proof as [102, Corollary 2.5] works), so $BS(n,-n)$ is RFD too.

In the case of $BS(n,n)$ the defining relation is a commutator, and we are in the situation of Theorem 7.7 part (ii). Thus $BS(n,n)$ is weak stabile, conditional on vanishing of the winding number invariant $w(ab^n a^{-1} b^{-n}, \phi)$ of Definition 7.2. On the other hand, in the case of $BS(n,-n)$, the defining relation is not a commutator, and we are in the situation of Theorem 7.7 part (i). Thus $BS(n,-n)$ is weakly stable. These results partly generalize the case of the fundamental groups of the torus and the Klein bottle covered in Theorem 1.5, which are $BS(1,1)$ and $BS(1,-1)$ respectively.

Note that the case of $BS(n,n)$ was discussed in [56, Theorem 4.10] where it is shown that these groups are not stable due to the non-vanishing of the associated winding number invariant; our result here provides a sort of converse. Compare also [56, Question 2 in Section 6].

Example 7.10. In [69, Theorem 11] Hadwin and Shulman show that if Γ is a one-relator group with non-trivial center and with abelianization Γ_{ab} that is not free abelian of rank two, then Γ is RFD. This gives another class of groups to which Theorem 7.7 may apply, at least once one has computed the homology: this should be possible (for example) using the explicit presentations in [115] and Smith normal form. We leave the details to the interested reader.

7.3 Three manifold groups

The following theorem specializes to Theorem 1.11 from the introduction.

Theorem 7.11. *Let Γ be the fundamental group of a closed (connected) aspherical three-manifold M, and assume that Γ is RFD. Write $H_2(M) = F \oplus T$ where F is a finitely generated free abelian group and T is a torsion group. Let $c_1, ..., c_m$ be a basis for F. Then Γ is weakly ucp stable, conditional on vanishing of $w(c_i, \phi)$ for all i.*

If moreover Γ satisfies the LLP of Definition 2.4, then we can drop the word "ucp" from the above.

Proof. As M is aspherical, it is a classifying space for Γ; in particular, M is a three-dimensional CW complex. As M is a manifold, its third homology group is $\mathbb{Z}$ (if M is orientable) or zero (if M is non-orientable); in either case, it is torsion-free. Moreover, as Γ has a finite-dimensional classifying space, it is torsion free. As noted in Remarks 5.6 and 6.2, Γ is UCT, and the BCK assembly map is an isomorphism. The result now follows from Theorem 7.4. □

Example 7.12. Perhaps the simplest class of groups covered by Theorem 7.11 are the fundamental groups of closed flat manifolds. For example, this class includes $\mathbb{Z}^3$, the fundamental group of the 3-torus. In general, there are ten such groups: see for example [142, Theorems 3.5.5 and 3.5.9] for the classification.

Fundamental groups of closed flat manifolds are amenable and residually finite, so are LLP and RFD (compare Remarks 5.7 and 5.5). Theorem 7.11 implies that the corresponding groups are weakly stable, conditional on vanishing of the winding number invariants coming from their two-homology. In particular, if the two-homology is torsion, then the corresponding group is weakly stable: this is the case for the fundamental group of the (non-orientable) manifold called $\mathcal{B}_4$ in [142, Theorem 3.5.9] as already pointed out by Dadarlat [43, Example 2.7]. It also follows for the (orientable) manifold M called $\mathcal{G}_6$[39] in [142, Theorems 3.5.5]: indeed, by [142, Corollary 3.5.10], we have $H_1(M) = \mathbb{Z}/4 \oplus \mathbb{Z}/4$, whence $H_2(M)$ is torsion by Poincaré duality.

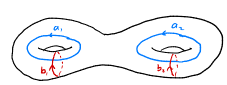
Note, however, that such groups are not stable, as pointed out in [56, Theorem 3.13]. In the case of $\Gamma = \mathbb{Z}^3$, this group is not even stable conditional on vanishing of the relevant winding number invariants: this follows from [67, Theorem 4.2]. It seems likely that this phenomenon – weak conditional stability holding, but conditional stability failing – is an issue for many other groups with three-dimensional classifying space.

We do not claim much originality in this flat manifold case – the result of Theorem 7.11 can be deduced from the same methods as [43, Theorem 1.5] for amenable groups – but hope that summarizing the main points here might be useful.

Thanks to Asaf Hadari for explaining some of the ideas in the following example to me.

Example 7.13. Let M be a closed, aspherical three manifold with $H_2(M)$ torsion. Then Theorem 7.11 implies that $\Gamma = \pi_1(M)$ is weakly ucp stable. To show that fairly direct computations are possible, here is a concrete construction of a three-manifold fibered over the circle, with non-amenable fundamental group, and such that $H_2(M)$ is torsion.

Let S be a closed orientable surface of genus $g = 2$ as pictured



with the curves $\{a_1, b_1, a_2, b_2\}$ as an ordered basis for $H_1(S) \cong \mathbb{Z}^4$. Note that the standard symplectic form (compare [63, Section 6.1.2]) with respect to this basis is given by

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Now, let $B = \left(\begin{smallmatrix} 5 & 3 \\ 3 & 2 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} B & 0 \\ 0 & B \end{smallmatrix}\right)$, which is an element of $M_4(\mathbb{Z})$ with determinant one such that $A^T J A = J$, i.e. A is an element of the sym-

[39] Also called the *Hantzche-Wendt manifold*.

plectic group $Sp(4,\mathbb{Z})$. Using [63, Theorem 6.4 (and Section 2.1 for the definition)], there is an orientation-preserving diffeomorphism $\phi_0 : S \to S$ that induces the map A on homology. Let moreover $\rho : S \to S$ be the left-right reflection around a plane separating the two holes as in the picture above, so that ρ induces the following map on homology

$$R = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let $\phi = \rho \circ \phi_0$, and let

$$M_\phi := S \times [0,1]/((x,0) \sim (\phi(x),1))$$

be the mapping torus of ϕ, which is a closed three manifold with (non-amenable) fundamental group $\Gamma = \pi_1(S) \rtimes \mathbb{Z}$, where the action of $\mathbb{Z}$ is via ϕ. Note that M_ϕ is aspherical by the same argument using the long exact sequence for a fibration that we used in the proof of Proposition 7.5; note moreover that Γ is RFD by [102, Theorem 2.8 (3)]. We claim that $H_2(M_\phi) = \mathbb{Z}/2$, whence Γ will be weakly ucp stable.

Indeed, as in [70, Example 2.8], the relevant part of the long exact homology sequence for a mapping torus looks like

$$\cdots \to H_2(S) \xrightarrow{\mathrm{id}-\phi_*} H_2(S) \to H_2(M_\phi) \to H_1(S) \xrightarrow{\mathrm{id}-\phi_*} H_1(S) \to \cdots .$$

As S is orientable, $H_2(S) \cong \mathbb{Z}$; as ϕ reverses orientation, the map $\phi_* : H_2(S) \to H_2(S)$ is multiplication by -1. On the other hand, the map $\phi_* : H_1(S) \to H_1(S)$ is given by the matrix RA; one readily checks that this matrix does not have one as an eigenvalue whence $\mathrm{id} - \phi_* : H_1(S) \to H_1(S)$ is injective. The result follows.

As recalled in Remark 5.5, the class of groups with the LLP is closed under semi-direct products with amenable groups, and therefore a group as in Example 7.13 would have the LLP if the corresponding surface group does. As pointed out in Example 7.8 above, this is open.

There are many other examples of three-manifolds with torsion, or even trivial, second homology to which Theorem 7.11 applies. For example, an interesting infinite family of closed hyperbolic three manifolds (which are aspherical as the universal cover is hyperbolic 3-space, and have RFD fundamental groups as in Remark 5.7) with the same homology as the 3-sphere are given in [24]. Note that the fundamental groups of these manifolds are perfect and non-amenable; Theorem 7.11 implies that they are all weakly stable.

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